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CONTENTS

On the Convergence of Certain Classes of Series of Functions. By R. D. CARMICHAEL, ... 77

On the Solution of Linear Equations in Infinitely Many Variables by Successive Approximations. By J. L. WALSH, ... 91

Self-Dual Plane Curves of the Fourth Order. By L. E. WEAR, . . 97

On the Groups of Isomorphisms of a System of Abelian Groups of Order p^m and Type $(n, 1, 1, \cdots, 1)$. By LOUIS C. MATHEWSON, 119

On the Satellite Line of the Cubic. By R. M. WINGER, ... 129

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On the Convergence of Certain Classes of Series of Functions.

By R. D. Carmichael.

§ 1. Introduction.

Let $v_n(x)$, $n = 0, 1, 2, \cdots$, be an infinite sequence of functions of x which may be written in the form

$$v_n(x) = \sum_{i=0}^{\mu_1} \sum_{j=0}^{\nu_1} a^{(n)}_{\mu-i,\, \nu-j} \; x^{\mu-i}(\log x)^{\nu-j} + \frac{M_n(x)}{x^{\mu_1-\mu+1}(\log x)^{\nu_1-\nu+1}} \tag{1}$$

where μ, μ_1, ν, ν_1 are integers such that $\mu_1 \geqq \mu$, $\nu_1 \geqq \nu$, and where $M_n(x)$, for fixed n, is a function of x which in absolute value is not greater than a constant M_n when $|x|$ is greater than some constant X and x is in a sector V formed by two rays proceeding from zero to infinity and including the positive axis of reals between them. Let $M_n(x)$, for every n, be an analytic function of x in every finite portion of the region of the x-plane just defined.

Moreover, let one of the coefficients a in (1), say $a^{(n)}_{kl}$ where k and l are not simultaneously zero, possess the following properties:

1) As n becomes infinite $a^{(n)}_{kl}$ becomes infinite, while its argument approaches a finite limit, say that it comes to coincide with the argument of a given constant σ, so that it may be written $a^{(n)}_{kl} = \sigma(\alpha^{(n)} + i\beta^{(n)})$ where $\alpha^{(n)}$ and $\beta^{(n)}$ are real. Suppose, moreover, that $\alpha^{(n)}$ is monotonic increasing when n is greater than some given constant. (It is easy to see that $\beta^{(n)}/\alpha^{(n)}$ approaches zero as n becomes infinite, a fact for which we shall have use later.)

2) The coefficient $a^{(n)}_{kl}$ has a dominance property of such sort that

$$\lim_{n=\infty} M_n(x)/a^{(n)}_{kl} = 0$$

for every x for which $M_n(x)$ is analytic and

$$\lim_{n=\infty} a^{(n)}_{ij}/a^{(n)}_{kl} = 0$$

unless simultaneously $i = k$ and $j = l$.

3) When $v_n(x)$ is one-termed so that it has the special value $v_n(x) = a^{(n)}_{kl} x^k (\log x)^l$ and $a^{(n)}_{kl}$ is real we make no further hypothesis; otherwise we

suppose that a positive constant ϵ_1 exists such that when n is greater than some appropriate N_{ϵ_1} we have $a^{(n+1)} - a^{(n)} \geqq \epsilon_1$.

Then let us consider the series of functions

$$S(x) = \sum_{n=0}^{\infty} c_n e^{v_n(x)} \tag{2}$$

where $c_0, c_1, c_2 \cdots$ are constants.

A point $\bar{x}$ will be called an exceptional or a non-exceptional point for the series $S(x)$ according as $\bar{x}$ is or is not a singularity of $v_n(x)$ for some value of n.

The principal object of the present paper is to consider the central convergence problem for the series $S(x)$ and for a certain other series defined in § 4 and having similar properties. It turns out that the convergence theory of these series may be readily developed in an elementary way. The character of the region of convergence and the uniform convergence of series $S(x)$ are treated in § 2. In § 3 theorems are established showing the coincidence in special cases of the regions of convergence of series $S(x)$ depending on different sequences $v_n(x)$. In § 4 are developed corresponding properties of a second class of series there defined. In § 5 cases are considered in which series of either class define functions having formal power series expansions.

A considerable variety of important classes of series are included in the general form (2). We shall exhibit a few of these.

(a) Let $v_n(x) = n \log x$. Our series is then an ascending power series. If we take $v_n(x) = -n \log x$ we have the descending power series.

(b) If we put $v_n(x) = -\lambda_n x$ where $\lambda_0, \lambda_1, \lambda_2, \cdots$ is an increasing sequence of real numbers tending to infinity then $S(x)$ is the generalized Dirichlet series. The extension of Dirichlet series employed in *Transactions American Mathematical Society* 17 (1916), p. 218, is also essentially included here.

(c) If we take $v_n(x) = \log \Gamma(x) - \log \Gamma(x+n)$, so that our series becomes the factorial series, it may be shown that $v_n(x)$ fulfills the requisite conditions, the asymptotic formula for $\Gamma(x)$ serving readily for this purpose.

(d) Indeed the more general class of series $\Sigma\, c_n g(x+n)$ which I have treated in *Trans. Amer. Math. Soc.* 17 (1916), pp. 207-232, may be shown to belong to the class treated here.

In the convergence proofs we shall have need of two lemmas which are reproduced here for the reader's convenience (for reference to the proofs of these lemmas see page 211 of the paper just cited):

Lemma I. *Let* $a_0 + a_1 + a_2 + \cdots$ *be a convergent series of constants and let* $\beta_0, \beta_1, \beta_2, \cdots$ *be an infinite sequence of numbers such that the series*

$\Sigma_n \mid \beta_{n+1} - \beta_n \mid$ is convergent. Then the series $a_0\beta_0 + a_1\beta_1 + a_2\beta_2 + \cdots$ is convergent.

Lemma II. *Let $a_0 + a_1 + a_2 + \cdots$ be a convergent series of constants and let $\beta_0, \beta_1, \beta_2, \cdots$ be an infinite sequence of functions of the complex variable x analytic in a given closed domain D and such that the series $\Sigma_n \mid \beta_{n+1} - \beta_n \mid$ converges uniformly in D. Then the series $a_0\beta_0 + a_1\beta_1 + a_2\beta_2 + \cdots$ converges uniformly in D.*

§ 2. *Character of the region of convergence of $S(x)$.*

Let us suppose that the series $S(x)$ converges for a given non-exceptional value x_0 of x, say briefly that $S(x_0)$ converges; and let us seek conditions on the non-exceptional value x_1 of x sufficient to ensure that $S(x_1)$ shall be convergent. We employ lemma I, taking

$$a_n = c_n\, e^{v_n(x_0)}, \quad \beta_n = e^{v_n(x_1) - v_n(x_0)}.$$

The series $\Sigma_n \mid \beta_{n+1} - \beta_n \mid$, whose convergence is sufficient to ensure the convergence of $S(x_1)$, may be put in the form

$$\sum_{n=N}^{\infty} \left| e^{a^{(n+1)}\tau} - e^{a^{(n)}\tau} \right| r_n \tag{3}$$

where

$$r_n = \frac{\left| e^{v_{n+1}(x_1) - v_{n+1}(x_0)} - e^{v_n(x_1) - v_n(x_0)} \right|}{\left| e^{a^{(n+1)}\tau} - e^{a^{(n)}\tau} \right|},$$

τ having the value

$$\tau = \theta\,\sigma\, \{x_1^k (\log x_1)^l - x_0^k (\log x_0)^l\},$$

where θ is a positive constant not greater than unity. We shall now suppose that the real part $R(\tau)$ of τ is negative.

We shall now show that a proper choice of θ will bring it about that r_n is bounded. If $v_n(x)$ is one-termed in the sense of condition 3) then we take $\theta = 1$, whence $r_n = 1$; otherwise we take θ to be less than unity. Then if we divide the numerator and the denominator of the fraction r_n by $|e^{a^{(n)}\tau}|$ we have left in the denominator a quantity bounded away from zero for every n in (3) is N is taken sufficiently large, as one sees from the condition in hypothesis 3). Moreover, each of the two exponential terms in the numerator of this fraction, and hence this numerator itself, approaches zero as n becomes infinite, as one sees readily through use of 1), 2) and 3) and

particularly of the dominance property of $a^{(n)}_{kl}$. Hence in any case r_n is bounded.*

It follows therefore that series (3) converges provided that the series

$$\sum_{n=N}^{\infty} \mid e^{a^{(n+1)}\tau} - e^{a^{(n)}\tau} \mid$$

is convergent.

To prove the convergence of this series we observe that

$$e^{a^{(n+1)}\tau} - e^{a^{(n)}\tau} = (1/\tau) \int_{a^{(n)}}^{a^{(n+1)}} e^{u\tau} du$$

so that

$$\mid e^{a^{(n+1)}\tau} - e^{a^{(n)}\tau} \mid \leqq (1/\mid \tau \mid) \int_{a^{(n)}}^{a^{(n+1)}} e^{uR(\tau)} du.$$

The series of which the nth term is the second member of this relation is obviously convergent since $a^{(n)}$ is ultimately monotonic and $R(\tau) < 0$ on account of the condition imposed on x_1.

Hence we have the part of the following theorem which refers to convergence (not absolute convergence):

Theorem I. *Let x_0 and x_1 be two values of x which are non-exceptional for the series $S(x)$ and suppose that $S(x_0)$ converges [converges absolutely]. Then $S(x_1)$ also converges [converges absolutely] provided that*

$$R\{\sigma x_1^k (\log x_1)^l\} < R\{\sigma x_0^k (\log x_0)^l\}.$$

The proof of the part of the theorem referring to absolute convergence is immediate. In fact it is sufficient to show that the ratio $e^{v_n(x_1)}/e^{v_n(x_0)}$ is bounded as n becomes infinite; and this is an immediate consequence of the hypotheses on $v_n(x)$.

By a *region C of convergence* of the series $S(x)$ we shall mean a region such that $S(x)$ converges for every non-exceptional value of x in the interior of C and diverges for every non-exceptional value of x exterior to C. In a similar way we define a *region Γ of absolute convergence of $S(x)$*.

By means of theorem I and the application of a classic method it is easy to determine the character of the regions of convergence and absolute convergence of $S(x)$. Compare the similar argument on p. 214 of the memoir already cited. We have the following result:

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* It should be observed that the only use of condition 3) in this proof is that made in showing that r_n is bounded, so that the theorem obtained is true when r_n is bounded even though condition 3) should not be satisfied.

Theorem II. *There exists a unique real number* λ $[\mu]$ *such that the region of convergence [absolute convergence] of the series* $S(x)$ *is bounded by the curve*

$$R\{\sigma x^k(\log x)^l\} = \lambda \quad [= \mu] \tag{4}$$

and lies on that side of this curve for which $R\{\sigma x^k(\log x)^l\}$ *is less than* λ $[\mu]$.

By the use of lemma II and a modification, mostly verbal in character, of the argument by which theorem I was established we may prove the following theorem:

Theorem III. *The series* $S(x)$ *converges uniformly in any closed domain* D *which lies within its region of convergence and contains no point which is exceptional for* $S(x)$ *or is a limit point of points which are exceptional for* $S(x)$.

As an immediate consequence of this we have the following:

Theorem IV. *The sum of the series* $S(x)$ *is a function* $S(x)$ *of* x *which is analytic at every non-exceptional point which is in the interior of its region of convergence and is not a limit point of exceptional points; and the derivatives of* $S(x)$ *at every such point may be found by differentiating the series* $S(x)$ *term by term.*

We should examine briefly the nature of the curves defined by equations of the form (4). For the case when $l = 0$ and k is positive I have already briefly described them in *Bull. Amer. Math. Soc.* (2) 23 (1917), pp. 424-425. In this case they are obviously algebraic. In particular, when $l = 0$ and $k = 1$ the curve is a straight line.

Suppose next that $l = 0$ and k is a negative integer $-t$. Then the curves have equations of the form $R(\sigma x^{-t}) = \eta_1$ and are again algebraic. If we write $\sigma = |\sigma| e^{i\phi}$, $\eta = \eta_1/|\sigma|$ and $x = re^{i\theta}$ where r is real and not negative, the equation of our curve may be written in polar coördinates r and θ in the form

$$\eta r^t = \cos(\phi - t\,\theta).$$

In case $\eta = 0$ our curve consists of $2t$ rays proceeding from 0 to ∞ and dividing the angular space about zero into $2t$ equal parts or sectors. The quantity $\cos(\phi - t\,\theta)$ is negative within alternate sectors of this set (the sectors of convergence) and positive within the others (in general the sectors of divergence). When η is not zero the curve consists of t branches lying within alternate sectors of the preceding set of $2t$ sectors, and in those for which $\cos(\phi - t\,\theta)$ is negative or positive according as η is negative or positive. For the case when $t = 1$ this curve is a circle.

When $k=0$, $l=1$ and σ is real equation (4) may be written in the form $R[\log x]=\eta_2$ or $|x|=\eta$, so that the curve is a circle about 0 as a center. As we have already seen, this includes the case when $S(x)$ is a power series either ascending or descending.

Except when $l=1$ and $k=0$, or $l=0$ and k is unrestricted, the curves (4) are in general transcendental.

§ 3. *Coincidence of the regions of convergence of different series $S(x)$.*

In connection with series $S(x)$ let us consider the two related series

$$S_{kl}(x)=\sum_{n=0}^{\infty} c_n e^{a_{kl}^{(n)} x^k (\log x)^l}, \qquad \bar{S}_{kl}(x)=\sum_{n=0}^{\infty} c_n e^{\sigma a^{(n)} x^k (\log x)^l}.$$

It is clear that each of these is a series of the general class $S(x)$ defined in § 1. Consequently the four theorems already established are valid for these series also. In the present section we establish relations among them and the more general series $S(x)$.

Theorem V. *The boundary curve of the region of convergence [absolute convergence] is the same for the three series $S(x)$, $S_{kl}(x)$, $\bar{S}_{kl}(x)$.*

It should be observed that the theorem says nothing about convergence [absolute convergence] on the boundary of the region of convergence [absolute convergence].

It is obviously sufficient to prove the theorem for the case of any two pairs of the three series, say for $S(x)$ and $\bar{S}_{kl}(x)$, and for $S_{kl}(x)$ and $\bar{S}_{kl}(x)$. In fact, it is enough to prove it for the first of these pairs. The method of proof is identical in the two cases, so that it is sufficient to carry out the work for either one of them alone. For the latter pair the formulae are somewhat simpler than for the former, so that we shall give in detail the proof of only that part of the theorem which relates to the series $S_{kl}(x)$ and $\bar{S}_{kl}(x)$. This proof falls into two parts.

1. Let $\bar{x}$ be any non-exceptional point in the interior of the region of convergence of $S_{kl}(x)$. We shall prove that $\bar{x}$ is likewise in the interior of the region of convergence of $\bar{S}_{kl}(x)$.

If $R\{\sigma x^k(\log x)^l\}=\lambda$ is the boundary of the region of convergence of $S_{kl}(x)$, then $R\{\sigma\bar{x}^k(\log\bar{x})^l\}<\lambda$. Then it is obvious that non-exceptional numbers x_0 and x_1 exist such that

$$R\{\sigma\bar{x}^k(\log\bar{x})^l\}<R\{\sigma x_1^k(\log x_1)^l\}<R\{\sigma x_0^k(\log x_0)^l\}<\lambda. \qquad (5)$$

Then x_0 and x_1 are points of convergence of $S_{kl}(x)$. Taking

$$a_n = c_n e^{a_{kl}^{(n)} x_0^k (\log x_0)^l}, \qquad \beta_n = e^{\sigma a^{(n)} x_1^k (\log x_1)^l - a_{kl}^{(n)} x_0^k (\log x_0)^l},$$

and applying lemma I we see that $\bar{S}_{kl}(x_1)$ converges if $\Sigma_n \mid \beta_{n+1} - \beta_n \mid$ converges. Now the exponent in the value of β_n may be written in the form

$$a^{(n)}\{\sigma x_1^k (\log x_1)^l - \sigma x_0^k (\log x_0)^l\} - \beta^{(n)} i\sigma x_0^k (\log x_0)^l.$$

We can now proceed by the method employed for a like matter near the beginning of § 2 and show that $\Sigma_n \mid \beta_{n+1} - \beta_n \mid$ converges. Hence $\bar{S}_{kl}(x_1)$ converges; and therefore $\bar{S}_{kl}(\bar{x})$ converges, as one sees through theorem I and the relation between $\bar{x}$ and x_1 in (5); and $\bar{x}$ is in fact in the interior of the region of convergence of $\bar{S}_{kl}(x)$.

2. If we suppose next that $\bar{x}$ is in the interior of the region of convergence of $\bar{S}_{kl}(x)$ it may be shown that it is likewise in the interior of the region of convergence of $S_{kl}(x)$. For this it is sufficient to apply lemma I as in the preceding case, taking this time for a_n and β_n the values

$$a_n = c_n e^{\sigma a^{(n)} x_0^k (\log x_0)^l}, \qquad \beta_n = e^{a_{kl}^{(n)} x_1^k (\log x_1)^l - \sigma a^{(n)} x_0^k (\log x_0)^l},$$

where x_0 and x_1 again satisfy relations (5), the curve $R\{\sigma x^k (\log x)^l\} = \lambda$ being now the boundary of the region of convergence of $\bar{S}_{kl}(x)$.

From the conclusions of the two preceding paragraphs we see that theorem V is true in so far as it relates to the region of convergence of $S_{kl}(x)$ and $\bar{S}_{kl}(x)$. The part relating to the region of absolute convergence of the same two series may be proved in the same way; or it may be proved more directly and more easily by a term-by-term comparison of the absolutely convergent series $S_{kl}(x_0)$ and $\bar{S}_{kl}(x_1)$ in the first case and $S_{kl}(x_1)$ and $\bar{S}_{kl}(x_0)$ in the second case, x_0 and x_1 being connected with an interior point $\bar{x}$ of the region of absolute convergence by a relation of the form (5), the curve $R\{\sigma x^k (\log x)^l\} = \lambda$ being now the boundary of the region of absolute convergence.

Theorem V thus established brings out the fact that for a given set of coefficients $c_0, c_1, c_2, \cdots$ in $S(x)$ the functions $v_n(x)$, subject to the permanent hypotheses as to character, can be modified in any way whatever so long as $a_{kl}^{(n)}$ is left unchanged (and indeed so long as σ and $a^{(n)}$ are left unchanged) without disturbing the position of the boundary curve of the region of convergence. Such changes may introduce or remove exceptional points or modify their character; but beyond this it affects convergence [absolute convergence] only on the boundary of the region.

Again, *Theorem V affords us a satisfying means of finding the convergence number* λ [*the absolute convergence number* μ] *of the series* $S(x)$. In

fact, it is the negative of the convergence abscissa [absolute convergence abscissa] of the Dirichlet series

$$D(t)=\sum_{n=0}^{\infty} c_n e^{-a^{(n)}t}$$

as one sees by comparison with $\bar{S}_{kl}(x)$ where $-t$ is thought of as replacing $\sigma x^k(\log x)^l$. Convenient formulae for the convergence abscissa of Dirichlet series are quoted or otherwise referred to on pages 224-225 of my memoir already cited.

§ 4. *Similar properties of a second class of series.*

Let us now consider similar questions for a series of the form

$$T(x)=c_0+\sum_{n=1}^{\infty} c_n P_1(x)P_2(x)\cdots P_n(x),$$

where $c_0, c_1, c_2, \cdots$ are constants and $P_1(x), P_2(x), P_3(x), \cdots$ a given sequence of functions.

A point $\bar{x}$ will be called exceptional for the series $T(x)$ if any one of the functions $P_n(x)$ has a singularity or a zero at $\bar{x}$; otherwise it will be called non-exceptional.

Let x_0 be a non-exceptional point such that $T(x_0)$ converges and let x_1 be a second non-exceptional point. Consider what relation between x_1 and x_0 is sufficient to ensure the convergence of the series $T(x_1)$. We employ lemma I, taking

$$a_n=c_nP_1(x_0)P_2(x_0)\cdots P_n(x_0),\quad \beta_n=\frac{P_1(x_1)\cdots P_1(x_1)}{P_1(x_0)\cdots P_n(x_0)}.$$

Then $T(x_1)$ converges if the series $\sum_n|\beta_{n+1}-\beta_n|$ converges. Now the ratio $R_n(x_1, x_0)$ of two consecutive terms of this series (the nth to the $(n-1)$th) may be put in the form

$$R_n(x_1,x_0)=\left|\frac{P_n(x_1)}{P_n(x_0)}\right|\cdot\frac{\left|\dfrac{P_{n+1}(x_1)}{P_{n+1}(x_0)}-1\right|}{\left|\dfrac{P_n(x_1)}{P_n(x_0)}-1\right|}.$$

Denote by $l(x_1, x_0)$ the greatest limit (the superior limit) of $R_n(x_1, x_0)$ as n becomes infinite.

Now if $l(x_1, x_0)$ depends explicitly upon x_1 and we determine x_1 so that $l(x_1, x_0)\leqq 1-\epsilon$, where ϵ is a positive quantity, we are assured that our series $\sum_n|\beta_{n+1}-\beta_n|$, and hence that our series $\mathrm{T}(x_1)$, converges. Moreover, if this inequality holds uniformly for x_1 in a given closed region containing no

exceptional points and no limit point of exceptional points either in its interior or on its boundary, then the series $T(x_1)$ converges uniformly in this region. Moreover, if $l(x_1, x_0)$ can be written in the form $\theta(x_1)/\theta(x_0)$, then it is easy to show that a number λ exists such that $\theta(x) = \lambda$ is the boundary of the region of convergence in the sense that $T(x)$ converges for every non-exceptional point for which $\theta(x) < \lambda$ and diverges for every non-exceptional point for which $\theta(x) > \lambda$. Again if $l(x_1, x_0)$ can be written in the form $\theta(x_1) - \theta(x_0) + 1$, we may likewise readily derive a similar result.

Now if $l(x_1, x_0)$ is independent of x_1 and x_0 and is greater than or equal to unity we get no information concerning the convergence of $T(x_1)$. But if $l(x_1, x_0)$ is independent of x_1 and x_0 and has a value less than unity we conclude that $T(x_1)$ is convergent without further restriction on x_1. Series of sort therefore have the interesting property that if they converge for a single non-exceptional value of x they converge for every non-exceptional value.

In one of the cases in which the foregoing argument fails, namely, that in which $\lim_{n=\infty} R_n(x_1, x_0) = 1$, we may profitably proceed to a consideration of the greatest limit of the quantity

$$n\{R_n(x_1, x_0) - 1\}.$$

We denote this greatest limit by $l_1(x_1, x_0)$. If it is less than or equal to $-1-\epsilon$ where ϵ is a positive constant, then the series $\Sigma_n \mid \beta_{n+1} - \beta_n \mid$ converges, whence we conclude that $T(x_1)$ also converges.

Now if $l_1(x_1, x_0)$ depends on x_1 and we choose x_1 so that $l_1(x_1, x_0) \leqq -1-\epsilon$ we have a situation similar to that just treated above and as before we can proceed readily to the determination of the character of the region of convergence, at least when $l_1(x_1, x_0)$ can be written as the quotient of a function of x_1, by a function of x_0, or $l_1(x_1, x_0) - 1$ as a difference of such functions.

Again, when $l_1(x_1, x_0)$ is independent of x_1 and x_0 we may treat the problem in the way indicated for the similar case above.

Out of other general criteria for the absolute convergence of series, as applied to the series $\Sigma_n \mid \beta_{n+1} - \beta_n \mid$, we may derive other related results. Those which we have stated have actually arisen frequently in special form in the investigation of the convergence of particular classes of series.*

It is desirable to examine certain special cases in which the foregoing

* One desiring to examine these special cases will find some of them treated and the others referred to in two papers of mine, namely, those in *Bull. Amer. Math. Soc.* (2) 8 (1917): 407-425 and Amer. Journ. Math., 36 (1914): 267-288, and in a paper by E. Cotton in *Bull. Soc. Math. Fr.*, 46 (1919): 69-84. The last paper is interesting for its general theorems, some of which are to be associated with the results of this section.

greatest limits exist in such way as to give rise to an elegant theory.

Let us suppose that

$$\lim_{n=\infty} \frac{P_n(x_1)}{P_n(x_0)}$$

exists, and let us denote its value by $m(x_1, x_0)$. Then if $m(x_1, x_0) \neq 1$ we have $\lim_{n=\infty} R_n(x_1, x_0) = | m(x_1, x_0) |$ so that $| m(x_1, x_0) |$ is to be identified with the $l(x_1, x_0)$ of the preceding discussion. An instance of this sort is afforded by a certain class of expansions in polynomials. Thus if we have

$$P_n(x) = a_{1n}(x-a) + a_{2n}(x-a)^2 + \cdots + a_{kn}(x-a)^k + \cdots a_{mn}(x-a)^m$$

where one of the coefficients a_{kn} dominates the others (in case there are any) in the sense that the quotient of any other by a_{kn} approaches zero as n becomes infinite, it is clear that $m(x_1, x_0) = (x_1 - a)^k/(x_0 - a)^k$. We see readily that the region of convergence is bounded by a circle about the point a as a center. For the special case of this in which $P_n(x) = x - a$ we have the usual ascending power series in $x - a$. By taking $P_n(x)$ a polynomial in $(x-a)^{-1}$ we obtain a like generalization of the usual descending power series in $x - a$.

This obviously may be further generalized by taking $P_n(x)$ in the form

$$P_n(x) = a_{1n}u_1(x) + \cdots + a_{kn}u_k(x) + \cdots + a_{mn}u_n(x)$$

where $u_1(x), \cdots, u_m(x)$ are given functions of x and the coefficient a_{kn} dominates the others in the same sense as before. The boundary of the region of convergence is now defined by an equation of the form $| u_k(x) | = \lambda$, and the region of convergence is on that side of this curve for which $| u_k(x) | < \lambda$. [It is clear that the finite series for $P_n(x)$ may be replaced by an infinite series if suitable hypotheses are made as to the character of its convergence.]

Let us now consider the case in which the foregoing limit value $m(x_1, x_0)$ is unity. Suppose that $P_n(x_1)/P_n(x_0)$ may be written in the form

$$\frac{P_n(x_1)}{P_n(x_0)} = 1 + \frac{m_1(x_1, x_0)}{n} + \frac{\xi_n(x_1, x_0)}{n^{1+\epsilon}},$$

where ϵ is a positive constant and $\xi_n(x_1, x_0)$ is bounded when n becomes infinite. Then we have

$$\lim_{n=\infty} n\{R(x_1, x_0) - 1\} = \lim_{n=\infty} n \left[\left| \frac{P_n(x_1)}{P_n(x_0)} \right| \cdot \left| \frac{\dfrac{m_1(x_1, x_0)}{n+1} + \dfrac{\xi_{n+1}(x_1, x_0)}{(n+1)^{1+\epsilon}}}{\dfrac{m_1(x_1, x_0)}{n} + \dfrac{\xi_{n+1}(x_1, x_0)}{n^{1+\epsilon}}} \right| - 1 \right]$$

$$= \lim_{n=\infty} n \left[\left\{ 1 + \frac{R\{m_1(x_1, x_0)\}}{n} \right\} \{1 - (1/n)\} - 1 \right]$$

$$= R\{m_1(x_1, x_0)\} - 1.$$

Hence $R\{m_1(x_1, x_0)\} - 1$ may be identified with the limit $l_1(x_1, x_0)$ of the preceding more general treatment; and the relevant convergence properties are therefore expressible in simple and elegant form.

It may be observed that it is essentially the Gauss criterion of convergence which lies at the bottom of the special result just obtained. Significant extensions of this result may be secured, if needed, by using a more delicate criterion than that of Gauss, say one of the infinite sequence of criteria due to Kummer (see *Encyclopédie des Sc. Math.*, I, p. 223).

The condition last indicated in detail is realized in the case of factorial series. Here we have $P_n(x) = 1/(x+n-1) = \Gamma(x+n-1)/\Gamma(x+n)$ so that

$$\frac{P_n(x_1)}{P_n(x_0)} = \frac{x_0+n-1}{x_1+n-1} = 1 + \frac{x_0 - x_1}{n} + \frac{\xi_n(x_1, x_0)}{n^2}:$$

Hence if the factorial series converges for a non-exceptional point x_0 it converges also for the non-exceptional point x_1 if $R(x_1) > R(x_0)$, as is well known.

It may be seen also that this condition is simply realized in a much more general class of cases. Let us put

$$P_n(x) = \frac{g(x+n)}{g(x+n-1)}$$

where $g(x)$ is a given function possessing the asymptotic expansion

$$g(x) \sim x^{\mu+\sigma x} e^{\alpha+\beta x}(1 + (a_1/x) + \cdots), \qquad \sigma \neq 0,$$

valid in a sector V formed by two rays proceeding from 0 to ∞ and including between them the positive axis of reals. Then we have as to n an asymptotic relation of the form

$$P_n(x) \sim n^{\sigma} e^{\sigma-\beta}(1 + (\mu + \sigma x)/n + \cdots);$$

whence

$$\frac{P_n(x_1)}{P_n(x_0)} \sim 1 + \frac{\sigma(x_1 - x_0)}{n} + \cdots$$

From this we conclude that our series $T(x_1)$ in this case converges if $R(\sigma x_1) < R(\sigma x_0)$, and that the boundary of its region of convergence is a straight line $R(\sigma x) = \lambda$.

There is another range of cases, of which the generalized Dirichlet series affords an example, in which one may readily conclude to the convergence of

$\Sigma_n \mid \beta_{n+1} - \beta_n \mid$ and hence of $T(x_1)$. Let us suppose that $\beta_{n+1} - \beta_n$ may be written in the form of an integral

$$\beta_{n+1} - \beta_n = \int_{C_n} u(t, x_1, x_0)\, dt$$

where C_n is a finite path of integration for each n, no two of these paths having a common arc. Let C be any path made up of all the paths C_l, C_{l+1} C_{l+2}, $\cdots$ (where l is a given integer) and any other paths which it is convenient to include. Then if the integral

$$\int_C \mid u(t, x_1, x_0) \mid dt$$

exists when x_1 is related in a specified way to x_0, it is clear that the series $\Sigma_n \mid \beta_{n+1} - \beta_n \mid$, and hence the series $T(x_1)$, converges under the same hypothesis as to the relation of x_1 and x_0.

If we take

$$P_1(x) = e^{-\lambda_1 x}, \quad P_n(x) = e^{-\lambda_n x} + \lambda_{n-1} x \text{ when } n > 1,$$

where $\lambda_1, \lambda_2, \lambda_3, \cdots$ is a monotone increasing sequence such that λ_n becomes infinite with n, our series $T(x)$ with $c_0 = 0$ is the Dirichlet series and the method indicated may be applied in the classic way illustrated already in § 2.

In the problem of representing functions with given properties or of finding functions with anticipated properties in given regions, it is sometimes desirable to have representations of them valid in certain preassigned regions of the plane. It is therefore of interest to ask under what simple conditions one will have a region of convergence of specified form: We add here a few remarks on this matter.

One of the simplest regions of convergence is a half plane bounded by a straight line. We have already observed certain cases in which the region of convergence is of this form, having the equation $R(\sigma x) = \lambda$. This suggests a more general case in which the same type of region of convergence arises. Let us suppose that the limit $m_1(x_1, x_0)$ of the preceding discussion is such that we have the relation

$$R\{m_1(x_1, x_0)\} = R\{\sigma(x_1 - x_0)\} \cdot \eta(x_1, x_0),$$

where $\eta(x_1, x_0)$ is positive whatever x_1 and x_0 are. Then it is easy to see that the region of convergence is bounded by a curve whose equation is of the form $R(\sigma x) = \lambda$.

From this special case it is clear that we may determine circumstances under which any one of a great variety of curves may be realized as the

boundary of the region of convergence and that we may attach this discussion to any one of the four limit quantities $l(x_1, x_0)$, $l_1(x_1, x_0)$, $m(x_1, x_0)$, $m_1(x_1, x_0)$ employed above. We have already seen in particular how circular regions may be realized in a great variety of instances. For other circular regions and half-plane regions, see my papers already cited and the papers referred to in them.

§ 5. *Cases in which the Series Define Functions Having Formal Power Series Expansions.*

Owing to the great importance, in the theory of differential and difference equations and in other parts of analysis, of functions possessing formal power series expansions either convergent or divergent, it is desirable to know the circumstances under which our series $T(x)$ and $S(x)$ can be formally transformed into power series and these conversely into series $T(x)$ or $S(x)$.

Let us consider the case of series $T(x)$ where $P_n(x)$ has the descending formal power series expansion

$$P_n(x) = \frac{c_{1n}}{x} + \frac{c_{2n}}{x^2} + \frac{c_{3n}}{x^3} + \cdots, \quad n = 1, 2, 3, \cdots, \tag{7}$$

where C_{1n} is different from zero for every value of n. It is obvious that the series $T(x)$ is then transformable formally into a descending power series in x and that the coefficients may be reckoned out by means of readily solvable recurrence relations.

A simple case of such a series $T(x)$ is the factorial series in which $P_n(x) = 1/(x + n - 1)$. The generalization of factorial series introduced in § 4 by aid of the function $g(x)$ also belongs here when $\sigma = -1$, as one may show without difficulty. If we should take σ to be the negative reciprocal of an integer k we should have a generalization to the case in which the formal descending power series in x are replaced by formal descending power series in the kth root of x.

If we have two functions $T_1(x)$ and $T_2(x)$ defined by two series $T(x)$ each depending on the set of functions (7) and if the product $T_1(x)T_2(x)$ is expansible into a series $T(x)$ of the same form, then the coefficients of the expansion may be found in the following manner: Transform the series for $T_1(x)$ and $T_2(x)$ into formal descending power series in x, take the product of these latter and transform it formally into a series $T(x)$; this series will represent the product $T_1(x)T_2(x)$. This process will certainly be valid at least when $T_1(x)$ and $T_2(x)$ are asymptotic to their formal power series representations and the functions P are such that no function has two representa-

tions in the form of a series $T(x)$. An instance of this sort is afforded by the series defined in § 4 in terms of $g(x)$, provided that $\sigma = -1$, as one sees from the results in AMER. JOURN. MATH. 39 (1917): 385-403. It is not difficult to see that the case just treated is an instance of series $S(x)$ as well as of series $T(x)$.

UNIVERSITY OF ILLINOIS,
September, 1919.

On the Solution of Linear Equations in Infinitely Many Variables by Successive Approximations.*

By J. L. Walsh.

In this paper we shall consider systems of equations of the type

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots = c_1 \ , \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots = c_2 \ , \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots = c_3 \ , \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \ , \end{array}\right\} \tag{1}$$

where the a_{ij} and c_i are given real or complex numbers and the x_i are to be determined. Systems of type (1) have been solved by various means,† including the method of successive approximations, but this method has been used chiefly for Hilbert space [i. e., the space of points $\{x_k\}$ for which $\sum_{k=1}^{\infty} | x_k |^2$ converges] with corresponding restrictions on the a_{ij} and c_i.‡ It is the object of the present paper to give a number of new conditions under which (1) can be solved by successive approximations; in particular, it is shown that if (1) has a non-vanishing normal determinant and if a simple transformation of the system is made, then the method of successive approximations can be used. The method of successive approximations is very convenient for numerical computation.

We shall use the method of successive approximations to prove the following theorem, which applies to a system of equations of type slightly less general than (1):

Theorem I. *If there exist positive constants C, M, and P such that the coefficients of the system*

$$\left.\begin{array}{r} x_1 + a_{12}x_2 + a_{13}x_3 + \cdots = c_1 \ , \\ x_2 + a_{23}x_3 + \cdots = c_2 \ , \\ x_3 + \cdots = c_3 \ , \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \ , \end{array}\right\} \tag{2}$$

satisfy the restrictions

* Presented to the American Mathematical Society (Chicago), Apr. 7, 1917.

† See, e. g., F. Riesz, *Équations Linéaires.*

‡ See E. Goldschmidt, Würzburg Dissertation (1912).

$|c_k| \leqq MC^k$, $\sum_{j=k+1}^{\infty} |a_{kj}|$ *convergent* $(k=1, 2, \cdots)$, $\sum_{j=k+1}^{\infty} |a_{kj}| \leqq P$ *for every* $k > K$, $C < (1/P)$, $C \leqq 1$, *then* (2) *has one solution and only one solution for which* $|x_k| \leqq \mu\gamma^k$, $\gamma < (1/P)$, $\gamma \leqq 1$.

We first consider the special case $K = 0$, and we take for approximations

$$\left.\begin{aligned} x_k^{(1)} &= c_k \quad (k=1, 2, \cdots), \\ x_k^{(i+1)} &= c_k - [a_{kk+1}x_{k+1}^{(i)} + a_{kk+2}x_{k+2}^{(i)} + \cdots] \quad (i=1, 2, \cdots). \end{aligned}\right\} \tag{3}$$

From (3) it follows that

$$\left.\begin{aligned} x_k^{(1)} &= c_k, \\ x_k^{(2)} - x_k^{(1)} &= \quad - [a_{k,\,k+1}x_{k+1}^{(1)} + a_{k,\,k+2}x_{k+2}^{(1)} + \cdots], \\ x_k^{(3)} - x_k^{(2)} &= \quad - [a_{k,\,k+1}(x_{k+1}^{(2)} - x_{k+1}^{(1)}) + a_{k,\,k+2}(x_{k+2}^{(2)} - x_{k+2}^{(1)}) + \cdots], \\ \cdot \quad \cdot \quad \cdot & \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad , \end{aligned}\right\} \tag{4}$$

and therefore

$$x_k = x_k^{(1)} + (x_k^{(2)} - x_k^{(1)}) + (x_k^{(3)} - x_k^{(2)}) + \cdots \tag{5}$$

$$<< MC^k + PMC^{k+1} + P^2MC^{k+2} + \cdots = MC^k/(1-PC). \tag{6}$$

The x_k as defined by (5) are a solution of (2), for if we add all the equations of (4) and sum by columns the resulting absolutely convergent double series, we have

$$x_k = c_k - [a_{kk+1}x_{k+1} + a_{kk+2}x_{k+2} + \cdots].$$

By (6) we see that for the x_k defined by (5) there exist μ and γ such that $|x_k| \leqq \mu\gamma^k$, $\gamma < (1/P)$, $\gamma \leqq 1$. Under this restriction the solution is unique, for if x_k' and x_k'' denote two solutions satisfying this restriction, their difference $\bar{x}_k = x_k' - x_k''$ is a solution of the homogeneous system corresponding to (2):

$$\left.\begin{aligned} \bar{x}_1 + a_{12}\bar{x}_2 + a_{13}\bar{x}_3 + \cdots &= 0, \\ \bar{x}_2 + a_{23}\bar{x}_3 + \cdots &= 0, \\ \bar{x}_3 + \cdots &= 0, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad &, \end{aligned}\right\} \tag{7}$$

and we have

$$|\bar{x}_k| \leqq NX^k, \ X < (1/P), \ X \leqq 1.$$

Place

$$\begin{aligned} x_k^{(1)} &= \bar{x}_k \quad (k=1, 2, \cdots), \\ x_k^{(i+1)} &= - [a_{kk+1}x_{k+1}^{(i)} + a_{kk+2}x_{k+2}^{(i)} + \cdots] \quad (i=1, 2, \cdots). \end{aligned}$$

from which it follows that $|x_k^{(i+1)}| \leqq P^i N X^{k+i}$. Hence

$$\lim_{i=\infty} x_k^{(i)} = 0.$$

But we have $x_k^{(i)} = x_k^{(1)} = x_k$ by equations (7). Hence $x_k = 0$, which proves the uniqueness of the solution and completes the proof of Theorem I for the case $K = 0$. The reader will easily complete the proof of Theorem I in its generality $[K \neq 0]$ by the use of mathematical induction. In this proof, it will appear that equations (3) and (5) will give x_k for every value of k.

When the x_k defined by (3) and (5) are computed in terms of the a_{ij} and c_i, it is found that

$$x_k = c_k - \sum_{j=k+1}^{\infty} a_{kj} c_j + \sum_{j=k+1}^{\infty} \sum_{i=j+1}^{\infty} a_{kj} a_{ji} c_i - \cdots, \tag{8}$$

which is the so-called Neumann series.

The following special case of Theorem I will be used in the sequel:

Theorem II. *If for the system* (2) *we have* $\sum_{j=k+1}^{\infty} |a_{kj}|$ *convergent for every* k, $\sum_{j=k+1}^{\infty} |a_{kj}| \leqq P < 1$ *for every* $k > K$,

$$|c_k| \leqq C \text{ for every } k,$$

then (2) *has one solution and only one solution for which the* x_k *are bounded. Moreover, this solution is given by the Neumann series* (8).*

We now return to the system of general type (1), and shall proceed to show that if (1) has a non-vanishing normal determinant and if the c_k are bounded, then (1) can be transformed into an equivalent system of type (2). This latter system will be shown to satisfy the hypotheses of Theorem II, and therefore the method of successive approximations can be used. To show the possibility of making this transformation we need the

Lemma. *In any non-vanishing determinant*

$$\Delta^{(k)} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{vmatrix},$$

* Theorem II is similar to a theorem given by von Koch, *Jahresbericht*, 1913, p. 289.

the rows can be arranged so that no minor

$$\Delta^{(i)} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1i} \\ a_{21} & a_{22} & \cdots & a_{2i} \\ . & . & . & . \\ . & . & . & . \\ a_{i1} & a_{i2} & \cdots & a_{ii} \end{vmatrix} \quad (i = 1, 2, \cdots k)$$

will vanish.

This lemma is evidently true (although trivial) for $k = 1$ and $k = 2$; the reader can readily complete the proof by induction.

We have supposed (1) to have a non-vanishing normal determinant; that is, we suppose $\sum\limits_{i,j=1}^{\infty} | a_{ij} - d_{ij} |$ to be convergent [d_{ij} is the Kronecker symbol whose value is zero or unity according as $i \neq j$ or $i = j$] and

$$\lim_{n=\infty} \Delta^{(n)} = \Delta \neq 0.$$

Then there exists k such that $\Delta^{(n)} \neq 0$ for $n = k, k + 1, \cdots$. Hence, by the Lemma, the order of the equations can be changed (if necessary) so that $\Delta^{(n)} \neq 0$ for $n = 1, 2, \cdots$. Such rearrangement will not affect the convergence of the double series $\sum\limits_{i,j=1}^{\infty} | a_{ij} - d_{ij} |$ nor the value of Δ.

We suppose, now, that this arrangement has been made, and we proceed to transform (1) into an equivalent system of the type

$$\left.\begin{aligned} b_{11}x_1 + b_{12}x_2 + b_{13}x_3 + \cdots &= \beta_1 \ , \\ b_{22}x_2 + b_{23}x_3 + \cdots &= \beta_2 \ , \\ b_{33}x_3 + \cdots &= \beta_3 \ , \\ . \quad . \quad . \quad . \quad . \quad . \end{aligned}\right\} \tag{9}$$

This transformation is made by placing* $b_{1k} = a_{1k}$, $\beta_1 = c_1$, and for $n > 1$,

$$b_{n,k} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,n-1} & a_{1k} \\ . & . & . & . & . \\ a_{n1} & a_{n2} & \cdots & a_{n,n-1} & a_{n,k} \end{vmatrix}, \quad (k = 1, 2, \cdots)$$

$$\beta_n = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,n-1} & c_1 \\ . & . & . & . & . \\ a_{n1} & a_{n2} & \cdots & a_{n,n-1} & c_n \end{vmatrix}.$$

The β_n as thus defined are bounded for if we set

* Riesz, *l. c.*, p. 11.

$$A_k = \sum_{i=1}^{\infty} | a_{ik} - d_{ik} |, \ \Pi = \prod_{k=1}^{\infty} (1 + A_k),$$

we shall have

$$| \beta_n | = \text{absolute value of} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,\,n-1} & c_1 \\ . & . & . & . & . \\ a_{n1} & a_{n2} & \cdots & a_{n,\,n-1} & c_n \end{vmatrix} \leqq \Pi \cdot C, \tag{10}$$

where $| c_k | \leqq C \ (k = 1, 2, \cdots)$.

Similarly, we shall prove that $\sum_{j=i+1}^{\infty} | b_{ij} |$ converges for every i. For by using a method similar to that used in (10), it follows that for $n < k$,

$$| b_{nk} | \leqq \Pi \left[| a_{1k} | + | a_{2k} | + \cdots + | a_{nk} | \right].$$

Hence we have

$$\begin{aligned} & | b_{n,\,n+1} | + | b_{n,\,n+2} | + \cdots \\ & \leqq \Pi \ [\ | a_{1,\,n+1} | + | a_{1,\,n+2} | + \cdots \\ & \qquad + | a_{2,\,n+1} | + | a_{2,\,n+2} | + \cdots \\ & \qquad + \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \\ & \qquad + | a_{n,\,n+1} | + | a_{n,\,n+2} | + \cdots] \ . \end{aligned}$$

Then for every value of n, the series of absolute values of the coefficients b_{nk}, $k > n$, converges; and its sum approaches zero as n becomes infinite.

None of the b_{nn} is zero and they approach a limit different from zero, so when we write system (9) in the form (2) the hypothesis of Theorem II is satisfied. Therefore we have shown that if (1) has a non-vanishing normal determinant and the c_i are bounded, and if (1) is transformed into an equivalent system of type (2), then the method of successive approximations can be used. Incidentally we have proved the theorem of von Koch that if (1) has a non-vanishing normal determinant and if the c_i are bounded, then (1) has one solution and only one solution such that the x_i are bounded.

We shall now state two theorems, each of which is proved precisely as Theorem I was proved. The x_k of each theorem are given by the Neumann series (8).

Theorem III. *If system* (2) *is such that* $\sum_{j=k+1}^{\infty} | a_{kj} |^p \ (p > 1)$ *converges for every* k, $\sum_{j=k+1}^{\infty} | a_{kj} |^p \leqq Q^p$ *for every* $k > K$,

$$| c_k | \leqq MC^k, \qquad C < \frac{1}{(1 + Q^{p/(p-1)})^{(p-1)/p}}$$

then (2) *has one solution and only one solution for which*

$$| x_k | \leqq \mu\gamma^k, \qquad \gamma < \frac{1}{(1+Q^{p/(p-1)})^{(p-1)/p}}. \ *$$

Theorem IV. *Under the restrictions* $| a_{ik} | \leqq NT^{k-i}$ *for every* $k > i$,

$$| c_k | \leqq MC^k, \quad TC < 1/(1+N),$$

system (2) *has one solution and only one solution for which*

$$| x_k | \leqq \mu\gamma^k, \quad T\gamma < 1/(1+N).\dagger$$

MADISON, WIS.
June, 1917.

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* A slightly less general theorem for the case $p = 2$ was proved by von Koch using infinite determinants. See *Proc. Camb. Cong. Math.* (1912) I, p. 354.

In proving Theorem III there will be found useful the following inequality due to Hölder:

$$\left| \sum_{k=1}^{\infty} a_k b_k \right|^p \leqq \left(\sum_{k=1}^{\infty} | a_k |^p \right) \left(\sum_{k=1}^{\infty} | b_k |^{\frac{p}{p-1}} \right)^{p-1}.$$

See Riesz, *l. c.*, p. 45.

† Cf. von Koch, *l. c.*, p. 355.

Self-Dual Plane Curves of the Fourth Order.

By L. E. Wear.

§ 1. *Introduction.*

The reciprocal, r^m, of a plane rational curve, ρ^n, may be regarded as obtained by a polarity which sends any point of ρ^n into a line of r^m and conversely. The singularities of ρ^n will go over into their dual singularities on r^m. Now the reciprocal is, in general, distinct from the point-curve and $m \neq n$. The question naturally rises as to when will the two curves coincide. Curves having this property may be called self-dual.* It is evident that for curves of this kind there must be a one-to-one correspondence between the singularities of ρ^n and r^m. In other words the order and class must be the same and we have a necessary condition for self-duality expressed by the equation,

$$n = n(n-1) - 2d - 3c, \text{ i.e. } 2d + 3c = n(n-2).$$

In this paper the quartic curve is to be considered and the equation becomes, for $n = 4$,

$$2d + 3c = 8.$$

There are two solutions for this equation, viz.:

$$\text{I.} \quad d = 1, \ c = 2$$
$$\text{II.} \quad d = 4, \ c = 0$$

These are, respectively, the limaçon and the degenerate case of two conics. They will be considered in this order.

Part I. The Limaçon.

§ 2. *The Equation of the curve.*

The curve is symmetrical with respect to an axis which cuts it in the double point and in two other points which are the vertices of the curve. If we take, as triangle of reference, the axis of the curve and the tangents at the vertices, then the equation is

$$x_0 = at^4 - (a+2)t^2, \ x_1 = (a-2)t^2 - a, \ x_2 = (a-1)t^3 - (a+1)t. \quad (1)$$

* Appel, in *Nouvelles Annales de Math.*, XIII, p. 207, calls such curves "auto-polaire." In that article he considered the problem of finding curves self-polar with regard to a given conic—the reverse of the present problem.

The Jacobians of these, two at a time, give the line equation of the curve, which is

$$\begin{aligned} \xi_0 &= (a-1)(a-2)\tau^2 - a(a+1), \\ \xi_1 &= a(a-1)\tau^4 - (a+1)(a+2)\tau^2, \\ \xi_2 &= 2a(2-a)\tau^3 + 2a(2+a)\tau. \end{aligned} \tag{2}$$

That equations (1) are the equations of a limaçon may be seen as follows:

The curve is reflected into itself in the axis $x_2 = 0$, the reflexion being effected by the transformation,

$$t + t' = 0,$$

of which $t = 0$, $t = \infty$ are the double elements. These are the vertices of the curve. In addition to these two points, the line $x_2 = 0$ cuts out the parameters $t^2 = (a+1)/(a-1)$. If these are substituted in the equations of the curve they give only one point, which must, then, be a double point of the curve.

The fundamental involution,* i. e. a pencil of binary forms apolar to each of (1), is

$$\begin{aligned} &(a-2)t^4 + 4(a-1)t^3 + 6at^2 + 4(a+1)t + (a+2) \\ &+ \lambda\,[(a-2)t^4 - 4(a-1)t^3 \\ &+ 6at^2 - 4(a+1)t + (a+2)] \end{aligned} \tag{3}$$

If we substitute the coefficients of this pencil in the condition for a cusp which is given by Professor Morley in his "Notes on Projective Geometry," p. 40, we find that the condition is satisfied, and, hence, that the curve has a cusp. Since the curve is reflected into itself in $x_2 = 0$, and since the cusp does not lie on the axis then there must be a second one, the reflexion of the first. The curve is, therefore, one having a double point and two cusps, i. e., is the limaçon.

The two cusps are given by $t^2 = 1$. The flexes are given by the Jacobian of the two members of the fundamental involution,† and are

$$t^2 = (a+1)(a+2)/(a-1)(a-2).$$

§ 3. *Polarities.*

Now any correlation which sends the curve into itself must interchange cusps and flexes. Hence there may be two such correlations corresponding to

* See a paper by Stahl, *Crelle*, Vol. 101, p. 300.

† Meyer: *Apolarität und Rationale Kurven*, p. 244.

the two ways in which the cusps and flexes may be paired. These two correlations are given by the transformations

$$t\tau = \sqrt{(a+1)(a+2)/(a-1)(a-2)} \tag{4}$$

and

$$t\tau = -\sqrt{(a+1)(a+2)/(a-1)(a-2)} \tag{5}$$

We require now that (4) and (5) shall send any point of the limaçon into a line of the curve and conversely. In order to do this operate with (4) or (5) on the equation of a point and identify the resulting expression in the parameter with the equation of a line. Thus will the parameter of a point of the curve be interchanged with that of a line of the curve and conversely, and hence there will be obtained a correlation which sends the curve into itself.

Now if we substitute in the incidence condition

$$(x\xi) = 0,$$

of point and line, the coördinates x_i from equations (1), we have the equation of any point of the limaçon; likewise, if we substitute the coördinates ξ_i from equations (2) we have the equation of any line tangent to the curve. Making these substitutions we find as the equations of point and line respectively,

$$\begin{aligned}&[at^4-(a+2)t^2]\,\xi_0+[(a-2)t^2-a]\,\xi_1\\&+[(a-1)t^3-(a+1)t]\,\xi_2=0\end{aligned} \tag{6}$$

and

$$\begin{aligned}&[(a-1)(a-2)\tau^2-a(a+1)]\,x_0+[a(a-1)\tau^4(a+1)(a+2)\tau^2]x_1\\&+[2a(2-a)\tau^3+2a(2+a)\tau]x_2=0\end{aligned} \tag{7}$$

If we now make the transformation (4) in equation (7), i. e. put $\tau = (1/t)\sqrt{(a+1)(a+2)/(a-1)(a-2)}$, clear the resulting equation of fractions and remove the factor $(a-1)$, the result is

$$\begin{aligned}&(a^2-1)(a-2)^2\sqrt{(a-1)(a-2)}\;[at^4-(a+2)t^2]x_0\\&+(a+1)^2(a+2)^2\sqrt{(a-1)(a-2)}\;[(a-2)t^2-a]x_1\\&-2a(a+2)(a-2)^2\sqrt{(a+1)(a+2)}\;[(a-1)t^3-(a+1)t]x_2=0.\end{aligned} \tag{8}$$

Since this is a line on the point t of the curve, it is identical with equation (6), regarding both equations as functions of the parameter. Making this identification, we have

$$\begin{aligned}-a\xi_0&=a(a^2-1)(a-2)^2\sqrt{(a-1)(a-2)}\;x_0,\\a\xi_1&=-a(a+1)^2(a+2)^2\sqrt{(a-1)(a-2)}\;x_1,\\(a+1)\xi_2&=2a(a+1)(a+2)(a-2)^2\sqrt{(a+1)(a+2)}\;x_2,\end{aligned}$$

wherein we have equated the coefficients of t^4, t^0 and t respectively. Dividing the first equation by $-a$, the second by a and the third by $a+1$, the equations become finally,

$$\left.\begin{aligned} \xi_0 &= (1-a^2)(a-2)^2\sqrt{(a-1)(a-2)}\ x_0, \\ \xi_1 &= -(a+1)^2(a+2)^2\sqrt{(a-1)(a-2)}\ x_1, \\ \xi_2 &= 2a(a+1)(a+2)(a-2)^2\sqrt{(a+1)(a+2)}\ x_2, \end{aligned}\right\} \tag{9}$$

If the transformation

$$\tau = -(1/t)\sqrt{(a+1)(a+2)/(a-1)(a-2)}$$

had been made in the equation of a line, then only ξ_2 will be changed in (9), since it comes from odd-powered terms, while ξ_0 and ξ_1 come from even powers. Furthermore the coördinate ξ_2 will be changed only in sign. Hence there arises from the second transformation the correlation

$$\left.\begin{aligned} \xi_0 &= (1-a^2)(a-2)^2\sqrt{(a-1)(a-2)}\ x_0, \\ \xi_1 &= -(a+1)^2(a+2)^2\sqrt{(a-1)(a-2)}\ x_1, \\ \xi_2 &= -2a(a+1)(a+2)(a-2)^2\sqrt{(a+1)(a+2)}\ x_2, \end{aligned}\right\} \tag{10}$$

Equations (9) and (10) are two correlations which send any point of the curve into a line of the curve, i. e. send the curve into itself. In particular dual singularities are interchanged, as may be easily verified.

Furthermore if we examine the determinants of equations (9) and (10) we find them to be of the form

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix},$$

i. e. the two correlations are actually polarities and the conics giving them are

$$\begin{aligned} &(1-a^2)(a-2)^2\sqrt{(a-1)(a-2)}\ x_0^2 \\ &-(a+1)^2(a+2)^2\sqrt{(a-1)(a-2)}\ x_1^2 \\ &\pm 2a(a+2)(a-2)^2\sqrt{(a+1)(a+2)}\ x_2^2 = 0. \end{aligned} \tag{11}$$

These conics are conjugate, in the sense that they have double contact and are reflected, the one into the other, by a pair of reflexions in the reference triangle. It follows that each conic is its own polar reciprocal as to the other. This fact is seen from a geometrical viewpoint, since a polarity leaving the curve unaltered must also leave the conic of the other polarity unaltered.

The points of contact of the two conics lie on the axis $x_2 = 0$. This can be proved from general considerations as follows: Call the vertices of the

limaçon A and A' (See Fig. 1); the polar of A ($t=\infty$) is the tangent at $A'(t=0)$ and conversely. Hence A and A' are a pair harmonic to the meets of $x_2=0$ with both of the conics. Also the polar of the double point D is the double line which cuts the axis at D', say. Then D and D' are also a pair of the involution on the line and the Jacobian of the two pairs (A, A') and (D, D') will give the points (C, C') where the conics cut $x_2=0$ and where they have contact,—since the polars of those points are tangent to both conics at the poles themselves. Further, the line $x_2=0$ is an axis of each one of the conics, since the polars of points on it, (A, A', D, D'), are lines perpendicular to the axis of the limaçon, and the pole of the latter is a point at infinity where any two of these perpendiculars intersect.

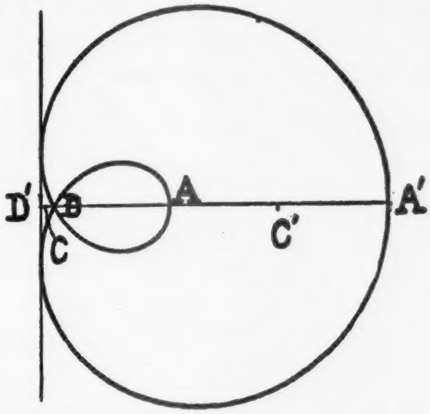


FIG. 1.

There will be certain points on the curve which will be fixed under the polarities, i. e. are transformed into tangents at the same points. These fixed points are four in number and are found by letting t and τ come together in equations (4) and (5). They are given by

$$t^4=(a+1)(a+2)/(a-1)(a-2).$$

Since at these four points the polar of t is a tangent to the curve at t, therefore each of the conics (11) has double contact with the limaçon. One conic has real contacts with the curve and the other has imaginary contacts.

§ 4. *Special Cases.*

Conics (11) will degenerate when their discriminants vanish. The latter are $\pm 2a(a-1)(a+1)^4(a-2)^5(a+2)^5\sqrt{(a+1)(a+2)}$ which vanish for the values $a=\pm 1, \pm 2$. But equations (11) vanish identically

for $a = -1$ and $a = +2$. Since the flexes are given by

$$t^2 = (a+1)(a+2)/(a-1)(a-2),$$

the value $a = +1$ signifies that two flexes have united at $t = \infty$ and it may be easily verified that they unite to form a third cusp. The cusp-tangent is $x_2 = 0$, which, taken twice, is the equation of the conics for the value $a = -1$. For $a = -2$, two flexes of the curve unite but in this case they form an undulation point, i. e. a point where the tangent to the curve meets the curve

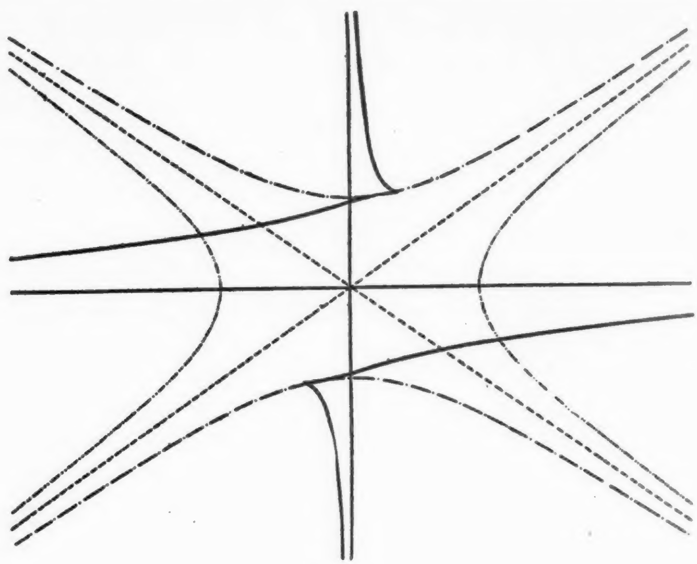


FIG. 2.

in four coincident points. The conics, for $a = -2$, degenerate into the line $x_0 = 0$, which is the equation of the undulation tangent.

§ 5. *Summary.*

We have seen that the limaçon admits of a reflexion given by the equation

$$t + t' = 0,$$

and is invariant under the two polarities π_0 and π_1. Call the reflexion R. Since $R^2 = 1$, the elements 1, R, form a group (G_2) of collineations under which the curve is invariant. It is evident also that

$$\pi_0 R = \pi_1 \text{ and } \pi_1 R = \pi_0.$$

Further

$$\pi_0\pi_1 = R \text{ and } \pi_0^2 = \pi_1^2 = 1.$$

Hence the

Theorem: *The limaçon is invariant under a G_4 consisting of two collineations and two polarities.*

Furthermore all possible polarities and correlations which leave the curve fixed are exhausted in π_0 and π_1. For, suppose another to exist—say π_m. Then either

$$\pi_0\pi_m = 1 \text{ or } \pi_0\pi_m = R$$

In the first case

$$\pi_m = \pi_0,$$

and in the second $\pi_m = \pi_1$.

In Figure 2 are shown the limaçon (in Cartesian coördinates), with the two conics, for the case $a = \frac{1}{2}$.

§ 6. *Satellite Conic.*

Of some interest in connecting with the plane quartic curve is the Satellite Conic of a line. In the case of the cubic curve the corresponding thing is the Satellite Line. Any line, ξ, will cut the cubic in three points. The tangents to the curve at these three points meet the curve again in three other points which lie on a line,* called the Satellite Line of ξ. In the case of the plane quartic a line, ξ, will cut the curve in four points T_i. The tangents to the curve at these four points meet the curve again in eight points which lie on a conic,† the Satellite Conic of the line ξ. The problem here is to find the actual equation of this conic for the limaçon.

The condition that three points of the plane rational quartic be on a line is as follows:

$$\begin{aligned} &p_{01}S_3^2 + p_{02}S_2S_3 + (p_{03} - p_{12})S_1S_3 + p_{12}S_2^2 + (p_{04} - p_{13})S_3 \\ &+ p_{13}S_1S_2 + (p_{14} - p_{23})S_2 + p_{23}S_1^2 + p_{24}S_1 + p_{34} = 0. \end{aligned}$$

The p_{ij} refer to the determinants $\begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix}$ of the fundamental involution $(at)^4 + \lambda(bt)^4$ (See Art. 2), and the S_i are symmetric functions of the three parameters of the points. Substituting the appropriate values of the p_{ij} from

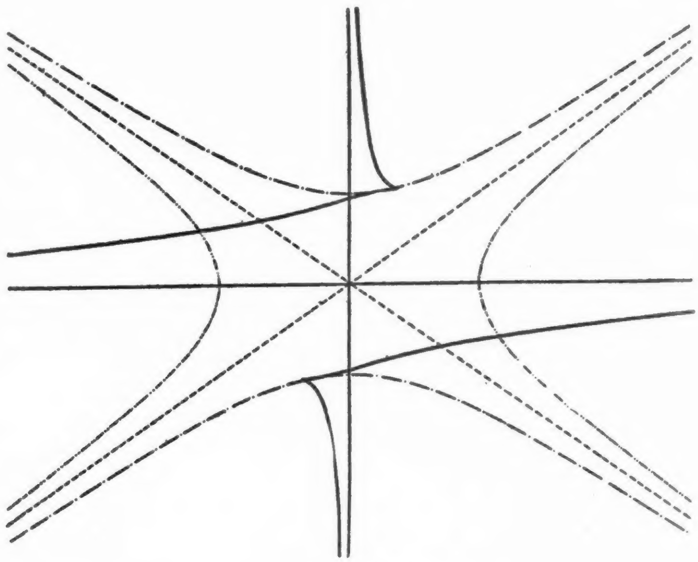
* Salmon, *Higher Plane Curves*, Art. 179.

† Salmon, *l. c.*, Art. 30.

for $a = -1$ and $a = +2$. Since the flexes are given by

$$t^2 = (a+1)(a+2)/(a-1)(a-2),$$

the value $a = +1$ signifies that two flexes have united at $t = \infty$ and it may be easily verified that they unite to form a third cusp. The cusp-tangent is $x_2 = 0$, which, taken twice, is the equation of the conics for the value $a = -1$. For $a = -2$, two flexes of the curve unite but in this case they form an undulation point, i. e. a point where the tangent to the curve meets the curve



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The condition that three points of the plane rational quartic be on a line is as follows:

$$\begin{aligned} p_{01}S_3^2 + p_{02}S_2S_3 + (p_{03} - p_{12})S_1S_3 + p_{12}S_2^2 + (p_{04} - p_{13})S_3 \\ + p_{13}S_1S_2 + (p_{14} - p_{23})S_2 + p_{23}S_1^2 + p_{24}S_1 + p_{34} = 0. \end{aligned}$$

The p_{ij} refer to the determinants $\begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix}$ of the fundamental involution $(at)^4 + \lambda(bt)^4$ (See Art. 2), and the S_i are symmetric functions of the three parameters of the points. Substituting the appropriate values of the p_{ij} from

* Salmon, *Higher Plane Curves*, Art. 179.

† Salmon, *l. c.*, Art. 30.

the fundamental involution of Art. 2, we have, as the condition that three points of the limaçon be collinear,

$$-(a-1)(a-2)S_3^2-2(a^2-a-1)S_1S_3+a(a-1)S_2^2$$
$$+2(a^2+a-1)S_2-a(a+1)S_1^2+(a+1)(a+2)=0.$$

Herein set $t_1=t_2=T$, $t_3=t$ and we have

$$\begin{aligned}&-(a-1)(a-2)t^2T^4-2(a^2-a-1)(t+2T)(tT^2)\\&+a(a-1)(2tT+T^2)^2+2(a^2+a-1)(2tT+T^2)\\&-a(a+1)(t+2T)^2+(a+1)(a+2)=0.\end{aligned} \tag{13}$$

This is a relation, $f(T^4, t^2)=0$, connecting T, the point of tangency of a line, and the two remaining points of intersection, t. It says that, given the line T, the two remaining points of intersection with the curve are given by (13). Conversely, given any point t on the curve, there are four tangents, T, to the curve from this point, given by (13). Since *any* line T, on one of the cusps will be incident with a t, then (T^2-1) must be a factor of equation (13). Rearranging the latter in powers of T we have

$$\begin{aligned}&[(a-1)(a-2)t^2-a(a-1)]T^4-4tT^3-2[(a^2-a+1)t^2\\&-(a^2+a+1)]T^2+4tT+[a(a+1)t^2-(a+1)(a+2)]=0,\end{aligned} \tag{13'}$$

and dividing by (T^2-1) we have, finally,

$$\begin{aligned}&[(a-1)(a-2)t^2-a(a-1)]T^2-4tT\\&-[a(a+1)t^2-(a+1)(a+2)]=0.\end{aligned} \tag{13''}$$

We have here a relation, $f(T^2, t^2)=0$, connecting the point t and the line T of the limaçon. For the self-dual quartic, then, on a line T are two points t, and from a point t are two tangents T, i. e. the relation is a perfectly symmetric one. It is invariant under the reflexion, $t+t'=0$, and also under the transformations (4) and (5). Letting $t=T$, we have a quartic giving the four points of the curve where a point t is coincident with a point of tangency, T. This quartic is

$$(a-1)(a-2)T^4-2(a^2+2)T^2+(a+1)(a+2)=0,$$

which factors into,

$$(T^2-1)[(a-1)(a-2)T^2-(a+1)(a+2)]=0.$$

These two factors give the cusps and flexes, respectively. Evidently at these points the tangent lines have three coincident intersections with the curve. The incidence condition of point and line is $(x\xi)=0$. Put therein the values

of x_i from equations (1), obtaining

$$\begin{aligned}&[aT^4 - (a+2)T^2]\,\xi_0 + [(a-2)T^2 - a]\,\xi_1 \\ &+ [(a-1)T^3 - (a+1)T]\,\xi_2 = 0.\end{aligned} \tag{14}$$

Arranging in powers of T we have

$$\begin{aligned}&a\xi_0 T^4 + (a-1)\xi_2 T^3 - [(a+2)\xi_0 - (a-2)\xi_1]T^2 \\ &- (a+1)\xi_2 T - a\xi_1 = 0.\end{aligned} \tag{14'}$$

Given a line ξ, equation (14′) fixes the parameters of the four points of intersection with the curve.

Now let the T's of (14′), be the same as those of (13″); i. e. let the two equations have common roots. The condition that the two have common roots is the vanishing of their eliminant, which will be of the fourth degree in the coefficients of (13″), and of the second degree in those of (14′).* Hence a $f(\xi^2, t^8) = 0$, a relation connecting ξ and the eight points t when the points of tangency T_i lie on ξ. This eliminant, formed according to Sylvester's dialytic method,† is

$$\begin{aligned}
&a[a(a+1)^2(a^4+4a^3-8a+4)\xi_0^2 + 2a(a+1)(a-1)(a-2)^2 \\
&(a^2+2a-2)\xi_0\xi_1 + a(a-1)^2(a-2)^4\xi_1^2 - (a+1)^3(a-1)^3 \\
&(a-2)\xi_2^2]t^8 - 8a(a+1)^2(a-1)(2a-1)\xi_0\xi_2 t^7 - 4[a(a+1)^2 \\
&(a+2)(a^4+3a^3-4a+2)\xi_0^2 + 2a(a-2) \\
&(a^6+a^5-5a^4-3a^3+7a^2-2a-1)\xi_0\xi_1 + a^2(a-1)^3(a-2)^3\xi_1^2 \\
&-(a+1)^2(a-1)^2(a^4-a^3-a^2-a+1)\xi_2^2]t^6 + 8[2(a+1)^2 \\
&(2a-1)(a^2+a-1)\xi_0\xi_2 + (a-1)^3(a-2)(2a+1)\xi_1\xi_2]t^5 \\
&+2[(a+1)^2(a+2)^2(3a^4+6a^3-4a+2)\xi_0^2 + 2(3a^8-21a^6 \\
&+46a^4-28a^2+8)\xi_0\xi_1 + (a-1)^2(a-2)^2(3a^4-6a^3+4a+2)\xi_1^2 \\
&-(a+1)^2(a-1)^2(3a^4-3a^2-4)\xi_2^2]t^4 - 8[(a+1)^3(a+2) \\
&(2a-1)\xi_0\xi_2 + 2(a-1)^2(2a+1)(a^2-a-1)\xi_1\xi_2]t^3 \\
&-4[a^2(a+1)^3(a+2)^3\xi_0^2 + 2a(a+2)(a^6-a^5-5a^4+3a^3+7a^2+2a-1) \\
&\xi_0\xi_1 + a(a-1)^2(a-2)(a^4-3a^3+4a+2)\xi_1^2 - (a+1)^2(a-1)^2 \\
&(a^4+a^3-a^2+a+1)\xi_2^2]t^2 + 8a(a+1)(a-1)^2(2a+1)\xi_1\xi_2 t \\
&+a[a(a+1)^2(a+2)^4\xi_0^2 + 2a(a+1)(a-1)(a+2)^2(a^2-2a-2) \\
&\xi_0\xi_1 + a(a-1)^2(a^4-4a^3+8a+4)\xi_1^2 - (a+1)^3(a-1)^3 \\
&(a+2)\xi_2^2] = 0.
\end{aligned} \tag{15}$$

Let a line ξ cut the curve in four points T. Draw the four tangents at these points. These tangents cut the curve in eight other points the para-

* See Salmon's Lessons on Higher Algebra, Art. 70.

† Salmon, l. c., Art. 83.

meters of which are given by the octavic (15). The next step is to find the equation of a conic passing through these eight points.

Any conic

$$\sum_{i,k=0}^{2} a_{ik}x_ix_k=0, \qquad a_{ik}=a_{ki}$$

will cut the curve in eight points which are obtained by substituting in the equation of the conic the values of x_i from equations (1). We have then

$$\begin{aligned}&a_{00}[at^4-(a+2)t^2]^2+a_{11}[(a-2)t^2-a]^2+a_{22}[(a-1)t^3\\&-(a+1)t]^2+2a_{01}[at^4-(a+2)t^2][(a-2)t_2-a]+2a_{02}[at^4\\&-(a+2)t^2][(a-1)t^3-(a+1)t]+2a_{12}[(a-2)t^2-a]\\&[(a-1)t^3-(a+1)t]=0.\end{aligned} \tag{16}$$

Simplifying and arranging in powers of t we have

$$\begin{aligned}&a^2a_{00}t^8+2a(a-1)a_{02}t^7+[-2a(a+2)a_{00}+(a-1)^2a_{22}\\&\qquad\qquad+2a(a-2)a_{01}]t^6\\&+[-2a(a+1)a_{02}-2(a-1)(a+2)a_{02}+2(a-1)(a-2)a_{12}]t^5\\&+[(a+2)^2a_{00}+(a-2)^2a_{11}-2(a+1)(a-1)a_{22}-4(a^2-2)a_{01}]t^4\\&+[-2(a+1)(a+2)a_{02}-4(a^2-a-1)a_{12}]t^3\\&+[-2a(a-2)a_{11}+(a+1)^2a_{22}+2a(a+2)a_{01}]t^2\\&+2a(a+1)a_{12}t+a^2a_{11}=0.\end{aligned} \tag{16'}$$

This octavic in t gives the parameters of the eight points cut out by any conic. By identifying (16′) with the octavic of equation (15), we have more than enough conditions to determine the coefficients a_{ik} of the conic. On identifying the two we have:

Coefficients of t^8,

$$\begin{aligned}aa_{00}=\;&a(a+1)^2(a^4+4a^3-8a+4)\xi_0^2+2a(a+1)(a-1)(a-2)^2\\&(a^2+2a-2)\xi_0\xi_1+a(a-1)^2(a-2)^4\xi_1^2-(a+1)^3(a-1)^3\\&(a-2)\xi_2^2;\end{aligned}$$

of t^7, $a_{02}=-4(a+1)^2(2a-1)\xi_0\xi_2$;

of t, $a_{12}=4(a-1)^2(2a+1)\xi_1\xi_2$;

of t^0,
$$\begin{aligned}aa_{11}=\;&a(a+1)^2(a+2)^4\xi_0^2+2a(a+1)(a-1)(a+2)^2\\&(a^2-2a-2)\xi_0\xi_1+a(a-1)^2(a^4-4a^3+8a+4)\xi_1^2\\&-(a+1)^3(a-1)^3(a+2)\xi_2^2;\end{aligned}$$

of t^4,
$$\begin{aligned}&-(a+1)(a-1)a_{22}-2(a^2-2)a_{01}=2(a+1)^2(a+2)^2\\&(a^4+2a^3+2a^2+4)\xi_0^2+4(a^8-5a^6+2a^4+18a^2-12)\xi_0\xi_1\\&+2(a-1)^2(a-2)^2(a^4-2a^3+2a^2-4)\xi_1^2\\&-2(a+1)^2(a-1)^2(a^4+a^2-4)\xi_2^2;\end{aligned} \tag{17}$$

of t^2, $(a+1)^2 a_{22} + 2a(a+2)a_{01} = -2a(a+1)^2(a+2)^3$
$$(a^2+2a+4)\xi_0^2 - 4a(a+2)(a^6-3a^4-4a^3+12a+6)\xi_0\xi_1 - 2a^4(a-1)^2(a-2)^2\xi_1^2 + 2(a+1)^2(a-1)^2 (a^4+2a^3+3a^2+2a-2)\xi_2^2 \tag{18}$$

By multiplying (17) by $(a+1)$ and (18) by $(a-1)$ and adding the resulting equations, we eliminate a_{22} and obtain

$$4a_{01} = 4(a+1)^2(a+2)^2(a^2+2a-2)\xi_0^2 + 8(a^6-9a^4+18a^2-6)\xi_0\xi_1 + 4(a-1)^2(a-2)^2(a^2-2a-2)\xi_1^2 - 4(a+1)^2(a-1)^2(a^2-3)\xi_2^2,$$

or

$$a_{01} = (a+1)^2(a+2)^2(a^2+2a-2)\xi_0^2 + 2(a^6-9a^4+18a^2-6)\xi_0\xi_1 + (a-1)^2(a-2)^2(a^2-2a-2)\xi_1^2 - (a+1)^2(a-1)^2(a^2-3)\xi_2^2.$$

Substituting this value of a_{01} in equation (18) and simplifying, we have

$$(a+1)^2 a_{22} = -4a(a+1)^4(a+2)^3\xi_0^2 - 8a^2(a+1)^2(a+2)(a-2)(a^2-3)\xi_0\xi_1 - 4a(a+1)^2(a-1)^2(a-2)^3\xi_1^2 + 4(a+1)^2(a-1)^2(a^4+2a^3-2a-1)\xi_2^2$$

Hence,

$$a_{22} = -4a(a+1)^2(a+2)^3\xi_0^2 - 8a^2(a+2)(a-2)(a^2-3)\xi_0\xi_1 - 4a(a-1)^2(a-2)^3\xi_1^2 + 4(a-1)^2(a^4+2a^3-2a-1)\xi_2^2.$$

All the coefficients of the conics are determined. By substituting the proper values in the coefficients of t^6 and t^5 the conditions arising from those two terms are found to be satisfied by the above values. Hence the coefficients of the conic are the following:

$$\begin{aligned}
aa_{00} &= a(a+1)^2(a^4+4a^3-8a+4)\xi_0^2 + 2a(a+1)(a-1)(a-2)^2 (a^2+2a-2)\xi_0\xi_1 + a(a-1)^2(a-2)^4\xi_1^2 - (a+1)^3(a-1)^3(a-2)\xi_2^2,\\
aa_{02} &= -4a(a+1)^2(2a-1)\xi_0\xi_2,\\
aa_{12} &= 4a(a-1)^2(2a+1)\xi_1\xi_2,\\
aa_{11} &= a(a+1)^2(a+2)^4\xi_0^2 + 2a(a+1)(a-1)(a+2)^2 (a^2-2a-2)\xi_0\xi_1 + a(a-1)^2(a^4-4a^3+8a+4)\xi_1^2 - (a+1)^3(a-1)^3(a+2)\xi_2^2,\\
aa_{01} &= a(a+1)^2(a+2)^2(a^2+2a-2)\xi_0^2 + 2a(a^6-9a^4+18a^2-6)\xi_0\xi_1 + a(a-1)^2(a-2)^2(a^2-2a-2)\xi_1^2 - a(a+1)^2 (a-1)^2(a^2-3)\xi_2^2,\\
aa_{22} &= -4a^2(a+1)^2(a+2)^3\xi_0^2 - 8a^3(a+2)(a-2)(a^2-3)\xi_0\xi_1 - 4a^2(a-1)^2(a-2)^3\xi_1^2 + 4a(a-1)^2(a^4+2a^3-2a-1)\xi_2^2.
\end{aligned}$$

The satellite conic is then as follows:

$$
\begin{aligned}
&[a(a+1)^2(a^4+4a^3-8a+4)\xi_0^2+2a(a+1)(a-1)(a-2)^2\\
&(a^2+2a-2)\xi_0\xi_1+a(a-1)^2(a-2)^4\xi_1^2-(a+1)^3(a-1)^3\\
&\qquad(a-2)\xi_2^2]x_0^2\\
&+[a(a+1)^2(a+2)^4\xi_0^2+2a(a+1)(a-1)(a+2)^2\\
&\qquad(a^2-2a-2)\xi_0\xi_1+a(a-1)^2(a^4-4a^3+8a+4)\xi_1^2\\
&\qquad-(a+1)^3(a-1)^3(a+2)\xi_2^2]x_1^2\\
&-4a[a(a+1)^2(a+2)^3\xi_0^2+2a^2(a+2)(a-2)(a^2-3)\xi_0\xi_1\\
&\qquad+a(a-1)^2(a-2)^3\xi_1^2-(a+1)^3(a-1)^3\xi_2^2]x_2^2\\
&+[8a(a-1)^2(2a+1)\xi_1\xi_2]x_1x_2-[8a(a+1)^2(2a-1)\xi_0\xi_2]x_0x_2\\
&+2[a(a+1)^2(a+2)^2(a^2+2a-2)\xi_0^2+2a(a^6-9a^4+18a^2-6)\\
&\qquad\xi_0\xi_1+a(a-1)^2(a-2)^2(a^2-2a-2)\xi_1^2-a(a+1)^2(a-1)^2\\
&\qquad(a^2-3)\xi_2^2]x_0x_1=0.
\end{aligned}
\tag{19}
$$

Given a line ξ, cutting the limaçon in four points T: the tangents at the points T cut in eight other points which lie on the conic given in equation (19).

Among interesting special cases is that of the line $x_2=0$, the axis of the curve. Here $\xi_0=0$, $\xi_1=0$, $\xi_2=1$. Substituting these values in (19) we have, after simplifying,

$$
\begin{aligned}
&(a+1)^2[(a+1)(a-1)(a-2)x_0^2+2a(a^2-3)x_0x_1+(a+1)\\
&\qquad(a-1)(a+2)x_1^2]-4a(a+1)^3(a-1)x_2^2=0.
\end{aligned}
\tag{20}
$$

This is a conic symmetrical as to the line $x_2=0$ and passing through the double point. The latter fact can be seen from the following considerations: The line $x_2=0$ cuts the curve at the vertices and at the double point. The two tangents at the double point have there three points in common with the curve. Hence, two of the eight points, t_i, being at the double point, the Satellite must pass through the latter.

Again the line joining the two cusps is $x_0-x_1=0$. Putting $\xi_0=1$, $\xi_1=-1$, $\xi_2=0$ in (19) and simplifying, we have

$$
\begin{aligned}
&(2a-1)^4x_0^2+2(16a^4-8a^2-1)x_0x_1+(2a+1)^4x_1^2\\
&\qquad\qquad-32a^2(2a+1)x_2^2=0.
\end{aligned}
\tag{21}
$$

This satellite of the line of cusps goes through the cusps themselves and through the two residual intersections of the cuspidal tangents.

Consider next the Satellite of a line ξ tangent to the curve. Such a line cuts in only three points T_i, the point of tangency and two others. The tangent lines at T_i will be the line ξ itself and the two tangents at other two points T_i. Call these tangents η and ζ. They cut the curve in t_1, t_2, and

t_1', t_2', respectively. Since ξ itself counts as a tangent at one of the points T_i, therefore the Satellite passes through the other points T_2, T_3. Since T_2, t_1, t_2 are on the conic, and likewise T_3, t_1', t_2', therefore the Satellite is composed of the two lines η and ζ. Hence, the Satellite conic of a line ξ tangent to the curve is composed of two lines, the tangents at the two points where ξ cuts the curve.

For example, the equation of the double line is

$$(a-2)^2x_0-(a+2)^2x_1=0.$$

Substituting $\xi_0=(a-2)^2$, $\xi_1=-(a+2)^2$, $\xi_2=0$ in equation (19), we find the Satellite of the double line to be

$$(a-2)^4x_0^2-2(a+2)^2(a-2)^2x_0x_1+(a+2)^4x_1^2=0, \tag{22}$$

or

$$[(a-2)^2x_0-(a+2)^2x_1]^2=0. \tag{22'}$$

That is, the Satellite of the double line is the double line itself taken twice.

The equation of the flex tangent at

$$T=\sqrt{(a+1)(a+2)/(a-1)(a-2)} \quad \text{is,}$$

$$(a+1)(a-1)(a-2)^2x_0+(a+1)^2(a+2)^2x_1-2a(a+2)(a-2)^2 \\ \sqrt{(a+1)(a+2)/a-1)(a-2)}\ x_2=0. \tag{23}$$

Setting $\xi_0=(a+1)(a-1)(a-2)^2$, $\xi_1=(a+1)^2(a+2)^2$,

$$\xi_2=-2a(a+2)(a-2)^2\sqrt{(a+1)(a+2)/(a-1)(a-2)}$$

in (19), we have, as the Satellite,

$$\begin{aligned}(a+1)^2&(a-1)(a-2)^4(a^2+2a-1)x_0^2+2(a+1)(a+2)^2\\ &(a-2)^2(a^4+4a^2-1)x_0x_1+(a+1)^2(a-1)(a+2)^4\\ &(a^2-2a-1)x_1^2-4a^2(a+1)(a+2)^3(a-2)^3x_2^2\\ &-4a\sqrt{(a+1)(a+2)/(a-1)(a-2)}\ [(a-1)(a+2)^3\\ &(a-2)^2(2a+1)x_1x_2-(a+1)(a+2)(a-2)^4\\ &(2a-1)x_0x_2]=0.\end{aligned} \tag{24}$$

In this case there is only one intersection in addition to the flex point, namely

$$t=-\frac{(a-1)}{(a+1)}\sqrt{\frac{(a+1)(a+2)}{(a-1)(a-2)}}.$$

The tangent at the latter point is

$$\begin{aligned}&(a+1)(a-2)^2(a^2+2a-1)x_0+(a-1)(a+2)^2(a^2-2a-1)x_1\\ &\quad+2a(a-1)(a+2)(a-2)^2\sqrt{(a+1)(a+2)/(a-1)(a-2)}\\ &\quad x_2=0.\end{aligned} \tag{25}$$

The Satellite (24) is the product of the flex tangent itself, and the line (25).

Let us consider now the dual idea. From any point x in the plane four tangent lines, t_i, can be drawn to the curve, touching at four points. From each of these four points are two tangents, T_i, to the curve. By a process exactly analogous to the preceding it can be shown that the eight lines, T_i, lie on a conic, the Satellite of x. The relation $f(x_i, \xi_i) = 0$ so found is identical with equation (19). To prove this statement it is sufficient to say that, owing to the self-duality of the curve, the relation $f(\xi_i, x_i) = 0$, connecting a line ξ and its Satellite must be identical with the relation $f(x_i, \xi_i) = 0$ connecting a point x and its Satellite, since the two ideas are dual ones. Hence equation (19) has a dual interpretation: Given ξ, cutting the curve in four points, T_i, it is the equation of a conic on the eight points where tangents at T_i meet the curve again; given x, from which are four tangents, t_i, to the curve, it is the equation of a conic on the eight tangent lines of the curve drawn from the points of contact of the lines t_i.

The center of reflexion, admitted by the curve, furnishes a good illustration. The coördinates of the center (0, 0, 1) substituted in (19) give the Satelllite

$$a(a+1)^2(a+2)^3\xi_0^2 + 2a^2(a+2)(a-2)(a^2-3)\xi_0\xi_1 \\ + a(a-1)^2(a-2)^3\xi_1^2 - (a+1)^3(a-1)^3\xi_2^2 = 0. \tag{26}$$

This conic is on the double line and the four tangents drawn from the two vertices of the curve.

From a point x on the curve are only two tangents to the curve (t_1, t_2) in addition to the tangent at x. The lines t_1, t_2 will touch at y and z, say. From each of the points y and z are two tangents to the curve which lie on the Satellite of x. Furthermore, since the point x is the point of contact of one of the tangents *from* x (the tangent *at* x) therefore t_1 and t_2 lie on the Satellite. Hence, from each of the points y and z are three tangents to the conic, which must, therefore, break up into the two points y and z. Thus, the Satellite of a point x on the curve is composed of two points,—the points of contact of tangents from x.

For example the Satellite of the double point, $[(a+1), -(a-1), 0]$, is

$$[(a+1)\xi_0 - (a-1)\xi_1]^2 = 0, \tag{27}$$

i. e., the square of the double point itself.

The coördinates of one of the cusps are 1, 1, 1. If substituted in (19), they give for the Satellite

$$\begin{aligned}(a+1)^2(a^2+2a-1)\xi_0^2+2(a^4+4a^2-1)\xi_0\xi_1+(a-1)^2\\(a^2-2a-1)\xi_1^2-(a+1)^2(a-1)^2\xi_2^2-2(a-1)^2\\(2a+1)\xi_1\xi_2+2(a+1)^2(2a-1)\xi_0\xi_2=0.\end{aligned} \tag{28}$$

From the cusp there is only one tangent to the curve (in addition to the cuspidal tangent) and this meets the curve at $t=-(a+1)/(a-1)$. The equation of the latter point is

$$\begin{aligned}(a+1)^2(a^2+2a-1)\xi_0+(a-1)^2(a^2-2a-1)\xi_1\\-(a+1)^2(a-1)^2\xi_2=0,\end{aligned} \tag{29}$$

and that of the cusp $(1, 1, 1)$, is

$$\xi_0+\xi_1+\xi_2=0. \tag{30}$$

Equations (29) and (30) multiplied together give (28), i. e. the Satellite of the cusp consists of two lines one of which is the cusp tangent.

PART II. TWO CONICS.

§ 7. *The G_{24} of a Four-Point.*

It was pointed out in the introduction that only two cases of self-dual quartics are possible and we come now to the second case, that of two conics regarded as a degenerate ρ^4.

To study the properties of the curve it is necessary to consider the four-point common to the two conics and the group of collineations connected therewith; since, if the pair of conics is to be unaltered by correlations then their common four-point and four-line must be merely interchanged. The pair of conics intersect in the same four points, after being acted upon by the correlations, as they did before. In this sense the common four-point and therefore the common self-conjugate triangle are fixed. We assume then that the two conics have a proper, common, self-conjugate triangle, which is taken as the triangle of reference, and that the four points are in the canonical form $(1, \pm 1, \pm 1)$.

The four-point is invariant under a G_{24} of collineations, consisting of reflexions and collineations of periods three and four. Call the four points by the numerals 1, 2, 3, 4 and indicate by subscripts the interchanges made by the transformations. E. g., the notation $C_{(ij)(k)(l)}$ means a collineation interchanging i and j and leaving k and l fixed. Then, in the first place, there is a set of four reflexions (including identity) in the reference triangle, i. e. leaving the vertices of the reference triangle for centers and the sides for axes. In the notation just explained they are:

Firstly, $\quad C_{(12)(34)},\ C_{(13)(24)},$ and $C_{(14)(23)}$.

These are of the type

$$x_0' = x_0, \quad x_1' = x_1, \quad x_2' = -x_2.$$

Secondly, there are reflexions

$$C_{(34)(1)(2)}, \ C_{(14)(2)(3)}, \ C_{(13)(2)(4)}, \ C_{(24)(1)(3)}, \ C_{(12)(3)(4)}, \ C_{(13)(2)(4)},$$

which interchange two of the points and leave the other two fixed. E. g.

$$x_0' = x_1, \quad x_1' = x_0, \quad x_2' = x_2,$$

the center of which is $(1, -1, 0,)$ and the axis $x_0 - x_1 = 0$.

Thirdly, eight collineations of period three, leaving one point fixed and interchanging the other three cyclically

$$C_{(123)(4)}, \ C_{(132)(4)}, \ C_{(134)(2)}, \ C_{(143)(2)}, \ C_{(124)(3)}, \ C_{(142)(3)}, \ C_{(234)(1)}, \ C_{(243)(1)}.$$

To illustrate, take the transformation

$$x_0' = x_2, \quad x_1' = x_0, \quad x_2' = x_1,$$

which is a member of the set.

Lastly, six collineations of period four, interchanging the four points cyclically

$$C_{(1234)}, \ C_{(1324)}, \ C_{(1342)}, \ C_{(1243)}, \ C_{(1423)}, \ C_{(1432)}.$$

E. g. $x_0' = -x_2$, $x_1' = -x_1$, $x_2' = x_0$, is a member of this set.

The twenty-four collineations form a G_{24}, under which the four-point is invariant. Consider now the effects of these transformations on any conic passing through the four points $(1, \pm 1, \pm 1)$. Such a conic may be taken in the form

$$a_0 x_0^2 + a_1 x_1^2 + a_2 x_2^2 = 0$$

provided that $$(a) = 0.$$

The reflexions 1, $C_{(ij)(kl)}$ involve only a change of signs and will, therefore, leave the conic unaltered. The other elements of the G_{24} in general send a conic on the four points into some other member of the pencil of conics determined by the four-point. Given, then, two conics considered as a quartic curve, the problem is to find correlations, and in particular, polarities, which will send any point of the curve into a line of the curve and conversely. This may be accomplished in either of two ways:

1°. The two conics may be interchanged under the correlations;

2°. Each conic may be unaltered.

§ 8. *General Case.*

Consider, first, the general case of two conics between which there exists no special relation. In general it is not possible to find a polarity that will leave each conic unaltered; but two conics related in a special manner do admit of such polarities and will be considered in the next article. It is necessary to look, then, for polarities that interchange the two conics.

Let

$$\alpha_0 x_0^2 + \alpha_1 x_1^2 + \alpha_2 x_2^2 = 0, \tag{31}$$

and

$$\beta_0 x_0^2 + \beta_1 x_1^2 + \beta_2 x_2^2 = 0, \tag{32}$$

where $(\alpha) = (\beta) = 0$, be two conics on the four points $(1, \pm 1, \pm 1)$. One of these is to be the polar reciprocal of the other as to some base conic. The latter may be taken in the form

$$a x_0^2 + b x_1^2 + c x_2^2 = 0; \tag{33}$$

since any polarity interchanging (31) and (32) will leave their common self-polar triangle unaltered and hence, either the latter is self-conjugate with respect to the base conic or else the base conic is tangent to two sides of the triangle, the third side being the chord joining the contacts. The latter possibility is considered in the next article. Now, if we require that the polarity as to (33) interchange (31) and (32) then the constants a, b, and c are determined to be

$$a : b : c := \pm\sqrt{\alpha_0\beta_0} \ : \ \pm\sqrt{\alpha_1\beta_1} \ : \ \pm\sqrt{\alpha_2\beta_2}$$

Take all possible combinations of signs and we obtain the following four polarities:

$$\begin{array}{llll}
\quad\ \pi_0 & \quad\ \pi_1 & \quad\ \pi_2 & \quad\ \pi_3 \\
\xi_0 = \sqrt{\alpha_0\beta_0}x_0, & = -\sqrt{\alpha_0\beta_0}x_0, & = \sqrt{\alpha_0\beta_0}x_0, & = \sqrt{\alpha_0\beta_0}x_0, \\
\xi_1 = \sqrt{\alpha_1\beta_1}x_1, & = \sqrt{\alpha_1\beta_1}x_1, & = -\sqrt{\alpha_1\beta_1}x_1, & = \sqrt{\alpha_1\beta_1}x_1, \\
\xi_2 = \sqrt{\alpha_2\beta_2}x_2, & = \sqrt{\alpha_2\beta_2}x_2, & = \sqrt{\alpha_2\beta_2}x_2, & = -\sqrt{\alpha_2\beta_2}x_2,
\end{array}$$

One of these, say π_0, having been obtained as above, then the others are given by the products $\pi_0 \cdot C_{(ij)(kl)}$. These are evidently correlations and they interchange the two conics, since π_0 interchanges them and $C_{(ij)(kl)}$ leave them fixed. That they are polarities appears from the fact that the products $\pi_0 C_{(mn)(pq)} \cdot \pi_0 C_{(rs)(tv)}$ which are collineations leaving each conic unaltered, must be contained in the set $C_{(ij)(kl)}$, and in particular $\pi_0 C_{(ij)(kl)} \cdot \pi_0 C_{(ij)(kl)}$ must be identity. Furthermore no other correlation could exist which would

transform the one conic into the other. For supposing such a one to exist, say π_m, then the product $\pi_0 \cdot \pi_m$ must be contained in the set $C_{(ij)(kl)}$, i. e.

$$\pi_0 \cdot \pi_m = C_{(mn)(pq)}$$

But

$$\pi_0 \cdot [\pi_0 \cdot C_{(mn)(pq)}] = C_{(mn)(pq)}$$

And hence

$$\pi_m = \pi_0 \cdot C_{(mn)(pq)}.^*$$

Hence the number of polarities is exhausted.

That there are just four polarities which interchange the conics is evident from a geometrical point of view. For, the four points of intersection of the conics must go into their common lines. It would appear then that twenty-four polarities are possible; but any polarity sending one of the four common points into one of the four common tangents carries a unique transformation of the three other points into the three other tangents. Hence there are only four polarities possible.†

The elements 1, $C_{(ij)(kl)}$ form a G_4 of collineations under which the quartic is invariant. Furthermore it was shown above that all correlations leaving the curve unaltered are included in the set π_0, $[\pi_0 \cdot C_{(ij)(kl)}]$ and that the products of the latter, two at a time give $C_{(ij)(kl)}$.

Hence the Theorem: *A quartic curve, composed of two conics, is invariant under a G_8, consisting of four collineations and four correlations.*

§ 9. *Two Conics Subject to the Condition $\Delta\theta_1^3 = \Delta_1\theta^3$.*

We come next to the case of two conics admitting not only the polarities of the preceding article but also a second kind, viz. those which leave each conic separately unaltered.

Assume a pair of conics on the four points $(1, \pm 1, \pm 1)$, and such that either one of them is reflected into the other by one of the collineations $C_{(ij)(kl)}$,—say by the collineation

$$x_0' = x_0, \quad x_1' = x_2, \quad x_2' = x_1,$$

Such a pair are

$$a_0 x_0^2 + a_1 x_1^2 + a_2 x_2^2 = 0, \tag{34}$$

$$a_0 x_0^2 + a_2 x_1^2 + a_1 x_2^2 = 0. \tag{35}$$

* See Weber, Lehrbuch der Algebra, Vol. II, p. 4.

† On relations between two conics see Clebsch, Léçons sur La Géométrie, Vol. I, p. 150 *et seq.*

Furthermore, since

$$C_{(lm)(pq)} \cdot C_{(lm)(p)(q)} = C_{(pq)(l)(m)},$$

then the conics will be reflected into each other by a second member of the set $C_{(ij)(k)(l)}$.

Just as in the preceding case the curve is unaltered by the group 1, $C_{(ij)(kl)}$, π_0, $\pi_0 \cdot C_{(ij)(kl)}$, the polarities being as follows:

$$\begin{array}{llll} \quad\pi_0 & \quad\pi_1 & \quad\pi_2 & \quad\pi_3 \\ \xi_0 = a_0x_0, & = -a_0x_0, & = a_0x_0, & = a_0x_0, \\ \xi_1 = \sqrt{a_1a_2}\, x_1, & = \sqrt{a_1a_2}\, x_1, & = -\sqrt{a_1a_2}\, x_1, & = \sqrt{a_1a_2}\, x_2, \\ \xi_2 = \sqrt{a_1a_2}\, x_2, & = \sqrt{a_1a_2}\, x_2, & = \sqrt{a_1a_2}\, x_2, & = -\sqrt{a_1a_2}\, x_2. \end{array}$$

Now the reflexions $C_{(ij)(k)(l)}$ have the effect of interchanging the two conics, and $C_{(ij)(kl)}$ leave each unaltered. Hence the effect of the products $C_{(ij)(k)(l)} \cdot C_{(ij)(kl)}$ is to interchange the two. By these products we obtain four collineations which send each conic into the other. The four include the two reflexions $C_{(ij)(k)(l)}$ and two of the set $C_{(ijkl)}$, i. e. collineations of period four. The eight elements form a collineation G_8 under which the curve is invariant. Add now a polarity π_0 which interchanges the pair of conics. The products $\pi_0 \cdot C_{(ij)(kl)}$, as before, interchange the pair of conics; but the products $\pi_0 \cdot C_{(ij)(k)(l)}$ and $\pi_0 \cdot C_{(ijkl)}$, leave each conic fixed. Of these the first two are polarities and the other two correlations of period four. The four correlations are as follows:

$$\begin{array}{llll} \multicolumn{2}{c}{\underline{\quad\pi_0 \cdot C_{(ij)(k)(l)}\quad}} & \multicolumn{2}{c}{\underline{\quad\pi_0 \cdot C_{(ijkl)},\quad}} \\ \xi_0 = a_0x_0, & = -a_0x_0, & = a_0x_0, & = a_0x_0, \\ \xi_1 = \sqrt{a_1a_2}\, x_1, & = \sqrt{a_1a_2}\, x_2, & = -\sqrt{a_1a_2}\, x_2, & = \sqrt{a_1a_2}\, x_2, \\ \xi_2 = \sqrt{a_1a_2}\, x_2, & = \sqrt{a_1a_2}\, x_1, & = \sqrt{a_1a_2}\, x_1, & = -\sqrt{a_1a_2}\, x_1. \end{array}$$

It is readily seen that the determinants of the first two are symmetrical and that those of the other two are not. Hence the former are polarities while the latter are correlations. That the correlations are of period four may be easily verified and is evident from the fact that they are $\pi_0 \cdot C_{(ijkl)}$ which, raised to the fourth power, is $\pi_0^4 \cdot C^4_{(ijkl)}$, i. e. identity. Furthermore there can be no other correlations leaving the pair of conics unaltered, a fact easily proved just as in the preceding case. Hence the curve is invariant under a G_{16} of collineations and correlations. Of the latter six are polarities and two are of period four.

Consider now the base conics of the polarities $\pi_0 \cdot C_{(ij)(k)(l)}$. Their equations are

$$a_0x_0^2 \pm 2\sqrt{a_1a_2}\ x_1x_2 = 0. \tag{36}$$

These are of the nature of conjugate hyperbolas and may be called conjugate conics. Furthermore each of the original pair is conjugate to both of (36). What we have then is this: Two conics (31) and (32) which admit of a reflexion, the one into the other; two conics (36) each of which is conjugate to the other and is also conjugate to both (31) and (32).

We inquire now as to whether or not it was necessary to assume conics that admit of one of the reflexions $C_{(ij)(k)(l)}$ in order to obtain polarities whose base conics are conjugate to the original pair. Two conics admitting a reflexion may be taken in the forms

$$ax_0^2 + bx_1^2 + cx_2^2 \pm 2gx_0x_2 = 0.$$

The invariants, using Salmon's notation (Conic Sections, p. 334), are as follows:

$$\Delta = abc - bg^2, \quad \Delta_1 = abc - bg^2, \quad \theta = (3abc - bg^2), \quad \theta_1 = (3abc - bg^2).$$

Since an invariant relation must be homogeneous and of the same degree in the coefficients of both forms, the only relation subsisting between the invariants is

$$\Delta\theta_1^3 = \Delta_1\theta^3.$$

We seek next the condition that two conics have a common conjugate conic and may admit, therefore, of polarities such as $\pi_0 \cdot C_{(ij)(k)(l)}$. The pencil of conics having double contact with $(x^2) = 0$, and with $(x\xi) = 0$ as the common chord of contact is

$$(x\xi)^2 - \lambda[(x^2)(\xi^2) - (x\xi)^2] = 0.\text{*} \qquad (37)$$

Values of λ that are equal but of opposite signs give conjugate pairs. For $\lambda = \pm 1$ we have

$$(x^2)(\xi^2) - 2(x\xi)^2 = 0 \qquad (38)$$

and $$(x^2) = 0 \qquad (39)$$

Now a second line, $$(x\eta) = 0,$$

will determine another pencil

$$(x\eta)^2 - \mu[(x^2)(\eta^2) - (x\eta)^2] = 0 \qquad (40)$$

and for $\mu = \pm 1$ the conjugate pair

$$(x^2)(\eta^2) - 2(x\eta)^2 = 0, \qquad (41)$$

* See Salmon's Conic Sections, p. 340.

$$(x^2) = 0. \tag{42}$$

Hence (38) and (41) are both conjugate to $(x^2) = 0$. Writing $\sigma_1 = (\xi^2)$ and $\sigma_2 = (\eta^2)$, the invariants are

$$\Delta = -\sigma_1^3, \quad \Delta_1 = -\sigma_2^3, \quad \theta = \sigma_1\lambda, \quad \theta_1 = \sigma_2\lambda,$$

where

$$\lambda = [\sigma_1\sigma_2 - 4(\xi_0\eta_0 + \xi_1\eta_1 + \xi_2\eta_2)^2].$$

The only invariantive relation subsisting between them is

$$\Delta\theta_1^3 = \Delta_1\theta^3.$$

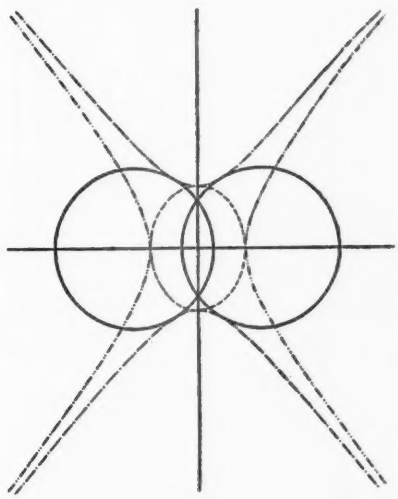

FIG. 3.

That is, the invariant relation that two conics admit of a reflexion the one into the other is identically the same as the condition that the two have a common conjugate conic. Hence the necessary and sufficient condition that polarities of the type $\pi_0 \cdot C_{(ij)(k)(l)}$ exist, is that the pair of conics admit of a reflexion and is expressed analytically by the above relation between the invariants.

We summarize this case by the Theorem: *Two conics characterized by the relation $\Delta\theta_1^3 = \Delta_1\theta^3$ are invariant under a G_{16} consisting of collineations and correlations. Of the latter six are polarities and two are of period four.*

In Fig. 3 the circles are the original pair of conics admitting a reflexion. Only three polarities are real, as shown in the figure.

§10. *The Clebschian Pair.*

We come finally to a special pair of conics which is invariant under the entire G_{24} of collineations and therefore, by adding a polarity and a G_{48} of collineations and correlations.

Suppose the conics of the preceding case are required to admit not only of two members of the set $C_{(ij)(k)(l)}$, as in that case, but also of a third member of the same set of collineations. Putting this further condition in the two conics we obtain the pair

$$x_0^2 + \omega x_1^2 + \omega^2 x_2^2 = 0, \quad x_0^2 + \omega^2 x_1^2 + \omega x_2^2 = 0. \tag{43}$$

These are either interchanged or left separately unaltered by the entire G_{24} of collineations. Hence, by adding a polarity, they are unaltered by a G_{48} of correlations and collineations. Of the former ten are polarities. Hence the Theorem: *The Clebschian pair of conics is invariant under a G_{48} of collineations and correlations. Of the latter ten are polarities and the others of periods three and four.*

On the Groups of Isomorphisms of a System of Abelian Groups of Order p^m and Type $(n, 1, 1, \cdots, 1)$.*

By Louis C. Mathewson.

Introduction.

Early in the study of groups of isomorphisms Moore showed that the group of isomorphisms of an abelian group of order p^m and type $(1, 1, \cdots, 1)$ is the linear homogenous group,† extensively discussed by Jordan in his *Traité des Substitutiones* (1870). Miller discussed the automorphisms of an abelian group of order p^m, type $(m-1, 1)$,‡ and later gave incidentally a formula for the order of the group of isomorphisms of any abelian group of order p^m.§ In 1907 Ranum through his study of the group of classes of congruent matrices showed that the group of isomorphisms of any given abelian group of order p^m was simply isomorphic with a certain chief n-ary linear congruence group.|| In the present paper the viewpoint is different and the groups are treated as abstract groups. The object is to study the groups of isomorphisms of the system of abelian groups of ordem p^m, type $(n, 1, \cdots, 1)$, $n > 1$, and to show that these groups of isomorphisms may be built upon the group of isomorphisms of an abelian group which contains no operators of order greater than p. To serve as a stepping stone to the general theory as well as to bring out the relations true for the first case, the case $n = 2$ will be considered immediately for $p = 2$ and for $p > 2$. In each development the group under consideration will be represented by G and its group of isomorphisms by I; p is used for an odd prime.

Theory.

Theorem 1. *The I of an abelian group of order* 2^{m+1}, *type* $(2, 1, \cdots, 1)$ *is of order* $2^m(2^m-2)(2^m-2^2)\cdots(2^m-2^{m-1})$ *and is simply isomorphic with a subgroup of index* 2^m-1 *in the holomorph of the abelian group of*

* Presented at the Dartmouth Meeting of the American Mathematical Society, Sept. 5, 1918.

† Cf. also Burnside, *Theory of Groups* (1897), §§ 171, 172 and Chap. XIV.

‡ Miller, *Transactions of the American Mathematical Society*, Vol. 2 (1901), pp. 259-264.

§ Miller, *Bulletin of the American Mathematical Society*, Vol. 20 (1913-14), p. 364.

|| Ranum, *Transactions of the American Mathematical Society*, Vol. 8 (1907), pp. 71-91.

order 2^m, type $(1, 1, \cdots, 1)$. This I may be obtained by extending an abelian group of order 2^m, type $(1, 1, \cdots, 1)$ by those operators from its own group of isomorphisms which leave one arbitrary operator in this abelian group fixed.

Suppose $m > 1$. The operators of order 2 in G evidently with the identity form a characteristic subgroup, H, of order 2^m. In H there is one (and only one) characteristic subgroup besides the identity. It is of order 2 and consists of the identity and the operator of order two which is the square of all the operators of order 4 in G. All the operators outside of H are of order 4. With the operators of H in identical correspondence any one of these operators may stand first, and an automorphism of G is then determined. Since these automorphisms are of order 2, commutative and number $2^m - 1$, the I of G contains an invariant abelian subgroup, H', of order 2^m, type $(1, 1, \cdots, 1)$. For, let $H \equiv 1, s_2, s_3, \cdots, s_h$, let $t^2 = s_2$, and let $G \equiv 1, s_2, s_3, \cdots, s_h, t, ts_2, \cdots, ts_h$ (all operators being commutative). Any operator ts_i, $i = 2, \cdots, h$, may correspond to t, so that the order of H' is h. Let $H' \equiv 1, v_2, v_3, \cdots, v_h$. Let the v that transforms t into ts_2 be v_2, into ts_3 be v_3, etc. Then, since each s is invariant under the v's

$$\begin{aligned} v_l^{-1} s_i v_l &= s_i, \\ v_l^{-1} t v_l &= t s_l. \end{aligned} \qquad l = 2, \cdots, h;\ i = 2, \cdots, h. \tag{1}$$

That the v's are of order 2 is evident from the fact that

$$v_l^{-1}(v_l^{-1} t v_l) v_l = v_l^{-1} t s_l v_l = v_l^{-1} t v_l \cdot v_l^{-1} s_l v_l = t s_l^2 = t;$$

and since all the operators of H' excepting the identity are of order 2, H' is abelian,* or,

$$\begin{aligned} v_a^{-1} v_b^{-1} t v_b v_a &= v_a^{-1} t s_b v_a = v_a^{-1} t v_a \cdot v_a^{-1} s_b v_a = t s_a s_b, \\ v_b^{-1} v_a^{-1} t v_a v_b &= v_b^{-1} t s_a v_b = v_b^{-1} t v_b \cdot v_b^{-1} s\ v_b = t s_b s_a, \end{aligned}$$

and since $t s_a s_b = t s_b s_a$, $v_a^{-1} v_b^{-1} t v_b v_a = v_b^{-1} v_a^{-1} t v_a v_b$, or $v_a v_b = v_b v_a$. That $v_a v_b$ transforms t into $t s_a s_b$ makes it possible to put H and H' into simple isomorphism in the following way: $s_i \sim v_i$, $i = 2, \cdots, h$.

From the nature of G, evidently H can be automorphic in all the ways an abelian group of order 2^m, type $(1, 1, \cdots, 1)$ can, except that s_2 must always correspond to itself. This means that one subgroup of order 2 in H is always fixed, so that the order of the quotient group of the I of G with respect to the invariant H' as a head is equal to the order of the group of isomorphisms of H divided by $2^m - 1$.*

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* Cf. Burnside, *loc. cit.*, p. 60.

* Burnside, *loc. cit.*, § 172.

It will now be shown that the I of G may be obtained by extending H' by operators which transform it in just the ways H may be transformed in G. This will be done by showing that an operator effecting any permissible automorphism of H, would produce a similar isomorphism among the operators of H'; i. e., if $u^{-1}s_hu = s_k$, then $u^{-1}v_hu = v_k$. It may be supposed that u is so chosen from the I of G that it transforms t into itself; for if not, u can be multiplied by such an operator from H' that the product will transform t into itself and at the same time effect exactly the same automorphism of H. Using (1),

$$(u^{-1}v_hu)^{-1}t(u^{-1}v_hu) = u^{-1}v_h^{-1}utu^{-1}v_hu = u^{-1}v_h^{-1}tv_hu = u^{-1}ts_hu = u^{-1}tu \cdot u^{-1}s_hu = ts_k, \text{just as } v_k^{-1}tv_k = ts_k; \qquad (2)$$

and since the v's are commutative with the s's and so also is $u^{-1}v_lu$ (because $(u^{-1}v_lu)^{-1}s_i(u^{-1}v_lu) = u^{-1}v_l^{-1}us_iu^{-1}v_lu = s_i$, for us_iu^{-1} is some s, and hence $v_l^{-1}us_iu^{-1}v_l = us_iu^{-1}$, and $u_l^{-1}us_iu^{-1}u = s_i$); therefore, $u^{-1}v_hu$ and v_k effect the same isomorphisms of G with itself. Thus, $u^{-1}v_hu = v_k$.

From the preceding it is obvious that the I of G is a subgroup of index $2^m - 1$ in the holomorph of the abelian group of order 2^m, type $(1, 1, \cdots, 1)$. This I should have exactly $\phi(4)$, or 2 invariant operators.* The operator besides the identity is easily shown to be v_2 according to the notation here used. (Note too that here v_2 corresponds to s_2 in an invariant subgroup of index 2^{m-1} in H). All the v's are commutative with v_2, and if in (2) $h = 2$ remembering that $u^{-1}s_2u = s_2$), the result from the end of the preceding paragraph is $u^{-1}v_2u = v_2$.

Next, the case in which p is an odd prime will be considered, and it will be shown that in this case the I of G is a direct product of two groups; and what these two groups are will be discussed.

The operators of order p in G evidently with the identity form a characteristic subgroup, J, of order p^m; also in J there is a characteristic subgroup, H, of order p whose $p - 1$ operators of order p are the pth powers of the operators of order p^2 in G. The operators of order p^2 correspond among themselves in every automorphism of G. With J in identical correspondence, any one of the p^m operators of order p^2 having the same pth power in H may stand first, and the automorphism of G is then fixed. Moreover, every such automorphism of G is of order p, and these $p^m - 1$ automorphisms of order p are commutative. These two facts may be proved just as similar facts were proved in the preceding case where the prime was 2. Let this invariant abelian subgroup of order p^m and type $(1, 1, \cdots, 1)$ in the I of G be E, and let its oper-

* Miller, Blichfeldt, and Dickson, *Finite Groups* (1916), p. 162.

ators be v's. If the generators of J are $s_1, s_2, \cdots, s_m$ where s_1 generates H (and $t^p = s_1$), then by a method analogous to that used for the even prime 2, it can be shown that the correspondence between J and E can be taken as $s_i \sim v_i$ $(i = 1, \cdots, m)$, where $v_i^{-1} t v_i = t s_i$ and where $v_i^{-1} s_l v_i = s_l$ $(i = 1, \cdots, m; l = 1, \cdots, m)$.

From the nature of G, evidently J can be automorphic in all the ways an abelian group of order p^m, type $(1, 1, \cdots, 1)$ can, excepting that H must always correspond to itself. Since J has $(p^m - 1)/(p - 1)$ subgroups of order p,* this means that the order of the quotient group of the I of G with respect to the invariant E is equal to the order of the group of isomorphisms of J divided by $(p^m - 1)/(p - 1)$, which gives the order of the I of G as $p^m(p - 1)(p^m - p)(p^m - p^2) \cdots\cdots (p^m - p^{m-1})$.

Consider those automorphisms of G in which the operators of H are in identical correspondence. Suppose that the operator, u, effecting the isomorphism under consideration transforms t into itself; for if not, u can be multiplied by such an operator from E that the product transforms t into itself and at the same time effects exactly the same automorphism of J. As in the preceding theorem, it can be shown easily that u transforms the operators of E among themselves in the same way it transforms the corresponding operators of J; that is, if

$$u^{-1} s_1 u = s_1, \quad u^{-1} s_j u = s_{j'}, \quad (j = 2, \cdots, m;\ j' = 2, \cdots, m);$$

then $u^{-1} v_1 u = v_1$, $u^{-1} v_j u = v_{j'}$, (j and j' have the same values respectively before).

Since all the automorphisms of G with the operators of the characteristic subgroup H in identical correspondence have been considered, the I of G evidently contains an invariant subgroup I' which is simply isomorphic with an abelian group of order p^m, type $(1, 1, \cdots, 1)$ extended by those operators from its own group of isomorphisms that leave the operators of one and only one of its subgroups of order p fixed. Since all other automorphisms of G arise from the automorphisms of H, and H is cyclic and of order p, obviously the quotient group of I with respect to I' is a cyclic group of order $p - 1$. It will now be shown that the I of G is simply isomorphic with the direct product of I' and the cyclic group of order $p - 1$.

The central of the I of G is the group of totitives (mod p^2) of order $\phi(p^2) = p(p - 1)$; that is, a cyclic group, the product of a cyclic group of order p by a cyclic group of order $p - 1$. Each of these operators in the central of the I of G transforms every operator of G into the same power of

* Burnside, *loc. cit.*, p. 59.

itself.* The cyclic group of order p in this central has been already obtained. In the notation here used, it is generated by v_1, since $v_1^{-1}tv_1 = ts_1 = t^{1+p}$, $v_1^{-2}tv_1^2 = ts_1^2 = t^{1+2p}$, etc.; and since, for any s, $v_1^{-1}s_iv_1 = s_i$ $(i = 1, \cdots, m)$, and the $(1+kp)$th power of s_i is s_i; and since $u^{-1}v_1u = v_1$. If now $s_1 \sim s_1^n$, then since $t^p = s_1$, t can $\sim t^n$, and the remainder of the automorphism of G may be set up by having the other generators (besides s_1) of J correspond to their own nth powers, $s_i \sim s_i^n$ $(i = 2, \cdots, m)$. If w effects this automorphism of G, $w^{-1}s_iw = s_i^n$ $(i = 1, \cdots, m)$, $w^{-1}tw = t^n$ (also $w^{-1}s_i^aw = s_i$, $w^{-1}t^aw = t$), it is necessary and sufficient to show that w is commutative with all the v's and with u. First, it will be shown that $w^{-1}v_iw = v_i$. Since $v_i^{-1}tv_i = ts_i$ $(i = 1, \cdots, m)$, $(w^{-1}v_iw)^{-1}t(w^{-1}v_iw) = w^{-1}v_i^{-1} \cdot wtw^{-1} \cdot v_iw$ $= w^{-1} \cdot v_i^{-1}t^av_i \cdot w = w^{-1}(v_i^{-1}tv_i)^aw = w^{-1}(ts_i)^aw = w^{-1}t^aw \cdot w^{-1}s_i^aw = ts_i$, just as $v_i^{-1}tv_i = ts_i$, and since the v's are commutative with s_j $(j = 1, \cdots, m)$ and so also is $w^{-1}v_iw$ (because $(w^{-1}v_iw)^{-1}s_j(w^{-1}v_iw) = w^{-1}v_i^{-1} \cdot ws_jw^{-1} \cdot v_iw$ $= w^{-1}v_i^{-1}s_j^av_iw = w^{-1}s_j^aw = s_j$); therefore, these two operators from I are identical or $w^{-1}v_iw = v_i$. Second, to show that w is commutative with u, use will be made of

$$\begin{cases} u^{-1}s_1u = s_1, \\ u^{-1}s_ju = s_{j'} \end{cases} \quad (j = 2, \cdots, m;\ j' = 2, \cdots, m) \text{ and } u^{-1}tu = t. \text{ Here}$$

$$(w^{-1}uw)^{-1}s_j(w^{-1}uw) = w^{-1}u^{-1} \cdot ws_jw^{-1} \cdot uw = w^{-1}u^{-1}s_j^auw \quad (j = 1, \cdots, m)$$
$$= \begin{cases} \text{if } j = 1,\ w^{-1}s_1^aw = s_1, \text{ just as } u^{-1}s_1u = s_1. \\ \text{if } j = 2, \cdots, m,\ w^{-1}s_{j'}^aw = s_{j'}, \text{ just as } u^{-1}s_ju = s_{j'}; \end{cases}$$
also $(w^{-1}uw)^{-1}t(w^{-1}uw) = w^{-1}u^{-1} \cdot wtw^{-1} \cdot uw = w^{-1}u^{-1}t^auw = w^{-1}t^aw = t$.

Hence, not only is the quotient group of I with respect to I' the cyclic group of order $p - 1$, but I contains such a cyclic group whose operators (excepting the identity) lie outside of I' and are commutative with each of the operators of I'. Therefore, I is simply isomorphic with the direct product of I' and the cyclic group of order $p - 1$, and for $m > 1$ there results the

Theorem 2. *The I of an abelian group of order p^{m+1}, type $(2, 1, \cdots, 1)$ is of order $p^m(p-1)(p^m - p) \cdots (p^m - p^{m-1})$ and is simply isomorphic with the direct product of a cyclic group of order $p - 1$ and the group formed by extending an abelian group of order p^m, type $(1, 1, \cdots, 1)$ by all those operators from its own group of isomorphisms which leave the operators of some one of its subgroups of order p in identical correspondence.*

As a side step from the main problem of this paper the following general

* Miller, *Transactions of the American Mathematical Society*, Vol. 2 (1901), p. 260.

proposition concerning a property obtaining in the abelian group just considered, will now be discussed.

Subsidiary Theorem. *If an abelian group G contains a characteristic subgroup H of prime order p, the I of G is simply isomorphic with a direct product, one factor of which is the cyclic group of order $p-1$. This I is then divisible if $p > 2$.*

Since "every abelian group is the direct product of its Sylow subgroups whenever its order is the product of more than one different prime," * H is in the Sylow subgroup, S, of order p^m and type $(m_1, m_2, \cdots, m_l)$, $m_1 \geqq m_2 \geqq \cdots \geqq m_l$, where $m = m_1 + m_2 + \cdots + m_l$ and $m_1 > 0$. If m_1 is not greater than m_2, there is no characteristic subgroup of order p. If $m_1 > m_2$, the group of order p in the cyclic group of order p^{m_1} is a characteristic subgroup of G (and the only characteristic subgroup of order p), since its operators are the p^{m_1-1}th powers of all the operators of S, its operators of order p being the p^{m_1-1}th powers of the operators of order p^{m_1} in S. Incidentally then, it has been shown that *a necessary and sufficient condition that an abelian group of order p^m contain a characteristic subgroup of order p is that there be one and only one largest invariant.* This group, H, of order p is a fundamental characteristic subgroup.† The remainder of the proof will now be worked out with respect to S, since the I of G is the direct product of the groups of isomorphisms of its Sylow subgroups.

The operators effecting the automorphisms of S in which the operators of H remain in identical correspondence form an invariant subgroup, I', of the group of isomorphisms of S (I_S). The quotient group of I_S with respect to I' is the group of isomorphisms of H; i. e., the cyclic group of order $p-1$. In the notation here used and with $p > 2$, the central of I_S is a cyclic group of order $\phi(p^{m_1}) = p^{m_1-1}(p-1)$, the product of a cyclic group of order p^{m_1-1} and another cyclic group of order $p-1$, and each of these operators in the central of I_S transforms every operator of S into the same power of itself.‡ The cyclic group of order p^{m_1-1} is in I', its operators being those which transform operators of order p^{m_1} in S into their $(1+kp)$th powers, $k = 1, 2, \cdots, p^{m_1-1}$. The operators of order p in H are invariant individually under such transformations, since these powers of each opeartor of order p in H are that operator itself. The operator of I_S which transform every operator of S into the 2nd, 3d, $\cdots$, $(p-1)$th powers of itself, evidently transform

* Miller, Blichfeldt, and Dickson, *loc. cit.*, p. 87.

† Cf. Miller, AMERICAN JOURNAL OF MATHEMATICS, Vol. XXVII (1905), p. 15; also Miller, Blichfeldt, and Dickson, *loc. cit.*, p. 110.

‡ Miller, *Transactions of the American Mathematical Society*, Vol. 1 (1900), p. 397; Vol. 2 (1901), p. 260.

the operators of H among themselves, so are not in I'. They with the identity, constitute a cyclic group of order $p-1$, and since they are in the central of I_S; they are individually commutative with the operators of I'. Hence, since the quotient group of I_S with respect to I' is a cyclic group of order $p-1$, and I_S contains a cyclic group of order $p-1$ having only the identity in common with I' and each of its operators is commutative with each of the operators of I', therefore I_S is simply isomorphic with the direct product of I' and a cyclic group of order $p-1$.

The theory of the second theorem will now be extended to any abelian group, G, of order p^{m+n-1}, type $(n, 1, 1, \cdots (m-1)$ units), $n > 1$, p a prime > 2. Let t be the operator of order p^n, and $s_2, \cdots, s_m$ independent generators of order p, and for convenience let $t^{p^{n-1}} = s_1$. From the observations under the Subsidiary Theorem, s_1 generates a characteristic cyclic subgroup, $H^{(n)}$, of order p. All the operators of order p in G form a characteristic abelian subgroup $H^{(n-1)}$ of order p^m, type $(1, \cdots, 1)$. $H^{(n)}$ is in $H^{(n-1)}$. Then there is a characteristic abelian subgroup, $H^{(n-2)}$, of order p^{m+1}, type $(2, 1, \cdots, 1)$ generated by the $p^{(n-2)}$th powers of operators of order p^n and $H^{(n-1)}$. So likewise, $H^{(n-3)}$ is generated by the p^{n-3}th powers of operators of order p^n and $H^{(n-1)}$; and in general, $H^{(n-r)}$, of order p^{m+r-1} and type $(r, 1, \cdots, 1)$, is generated by the p^{n-r}th powers of the operators of order p^n and $H^{(n-1)}$, $r = 2, \cdots, n-1$. Each of these characteristic subgroups contains the preceding, and the largest $(H^{(1)})$ is of index p in G itself. This series of subgroups forms a characteristic series of G.*

Now with the operators of $H^{(1)}$ in identical correspondence, evidently t can correspond to any one of $p^m - 1$ operators besides itself; it can correspond to itself multiplied by any one of the operators of $H^{(n-1)}$, since these products (and t) alone are of order p^n and have the same pth power that t has in $H^{(1)}$. As was shown in connection with Theorem 2, these isomorphisms (excepting the identity) are of order p and commutative. Hence, the I of G contains an invariant abelian subgroup, E, of order p^m, type $(1, \cdots, 1)$, which is simply isomorphic with $H^{(n-1)}$; and, moreover, if the same convenient notation be employed here as in the theorem to which reference has just been made, the correspondence, $v_i \sim s_i$, $i = 1, \cdots, m$, can be set up, where the v's are the independent generators of the subgroup E. The transformations, accordingly, are $v_i^{-1} t v_i = t s_i$, $(i = 1, \cdots, m)$, $v_i^{-1} s_j v_i = s_j$, $(i = 1, \cdots, m; j = 1, \cdots, m)$.

Now, let the operators of $H^{(2)}$ be in identical correspondence. t^p can

* Frobenius, *Berliner Sitzungsberichte* (1895), p. 1027; cf. Burnside, *loc. cit.*, §§ 163, 164.

correspond to $t^{p^{n-1}+p}$, $t^{2p^{n-1}+p}$, $\cdots$, $t^{(p-1)p^{n-1}+p}$ (or $t^p s_1$, $t^p s_1^2$, $\cdots$, $t^p s_1^{p-1}$, respectively, since $t^{p^{n-1}} = s_1$) besides itself, because these and these alone have the same pth power in $H^{(2)}$ and at the same time are themselves the pth powers of operators outside $H^{(1)}$. Simultaneously with $t^p \sim t^{p^{n-1}+p}$, t must correspond to some operator of order p^n whose pth power is $t^{p^{n-1}+p}$; such operators are $t^{p^{n-2}+1}$ times operators from $H^{(n-1)}$. The operators from $H^{(n-1)}$ may be supposed to be in $H^{(n)}$ also, for if not, the operator effecting the automorphism of G under consideration can be multiplied by such a v that $H^{(1)}$ is transformed as stated in the preceding and$t$ corresponds to $t^{p^{n-2}+1}$ times some operator from $H^{(n)}$. Accordingly, this isomorphism of G may be said to be effected by an operator which transforms every operator into its $(kp^{n-2}+1)$th powers, $k = 1, \cdots, p-1$.

Similarly, if the operators of $H^{(3)}$ are in identical correspondence, the additional isomorphisms of G spring from those in which the operators correspond to their $(kp^{n-3}+1)$th powers, $k = 1, \cdots, p-1$.

More generally, if the operators of $H^{(r)}$, $r = 2, \cdots, n-1$, are in identical isomorphism, $t^{p^{r-1}}$ can correspond to $t^{p^{n-1}+p^{r-1}}$, $t^{2p^{n-1}+p^{r-1}}$, $\ldots$, $t^{(p-1)p^{n-1}+p^{r-1}}$ (or $t^{p^{r-1}}s_1$, $\cdots$, $t^{p^{r-1}}s_1^{p-1}$, respectively) besides itself, because these operators and these only have the same pth power in $H^{(r)}$ and are themselves the p^{r-1}th powers of operators outside $H^{(1)}$. These isomorphisms are those effected by an operator which transforms the operators of G into their $(kp^{n-r}+1)$th powers, $k = 1, \cdots, p-1$. These automorphisms are p^{n-2} in number (because $r = 2, \cdots, n-1$ and $k = 1, \cdots, p-1$). If when $r = 1$, v_1 is included, these isomorphisms number p^{n-1}, and they are, moreover, those in which the operators of G go over into their $(1+kp)$th powers, $k = 1, 2, \cdots, p^{n-1}$. The only other powers are the 1st, 2nd, $\cdots$, $(p-1)$th, and when these are effected the characteristic subgroup $H^{(n)}$ takes all its automorphisms. But the operators effecting the possible transformations of all the operators into their same powers constitute the central of the I of G, a cyclic group of order $\phi(p^n) = p^{n-1}(p-1)$, (because the highest order of operators in G is p^n), the product of two cyclic groups, one of order $p-1$ and one of order p^{n-1}. From the Subsidiary Theorem the I of G is the direct product of this cyclic group of order $p-1$ and another subgroup, I'. The cyclic group of order p^{n-1} must be in I'. Suppose u to be employed to represent a generator of this cyclic group, F, of order p^{n-1}, so that $u^{-1}tu = t^{1+kp}$, $k = 1, \cdots, p^{n-1}$. With the operators of $H^{(n-1)}$ in identical correspondence, all the possible isomorphisms of G are effected by u and the v's. They all are commutative (since u is in the central of I) and the cross-cut of E and F is the cyclic group of order p generated by $v_1 (= u^{p-2})$.

Hence, I' contains an invariant abelian subgroup of order p^{m+n-2}, type $(n-1, 1, 1, \cdots, 1)$.

Finally, if the operators of $H^{(n)}$ alone are in identical correspondence, the remaining operators of $H^{(n-1)}$ have exactly the automorphisms of an abelian group of order p^{m-1}, type $(1, 1, \cdots, 1)$ when the operators of some one subgroup of order p remain fixed. If w effects such an automorphism (and leaves t invariant), it can be shown just as in Theorem 2 that w is commutative with v_1 but transforms the other operators of E just as the operators of $H^{(n-1)}$ outside of $H^{(n)}$ are transformed, and w, furthermore, would be found to be commutative with u. The number of the isomorphisms effected by w's would be $(p^m - p)(p^m - p^2) \cdot\cdot\cdot\cdot\cdot (p^m - p^{m-1})$.*

The following may be stated as a summary of these results:

Theorem 3. *The I of an abelian group G of order p^{m+n-1}, type $(n, 1, 1, \cdots, 1)$, p a prime > 2, $n > 2$, is of order $(p-1)p^{m+n-2}(p^m - p)(p^m - p^2) \cdot\cdot\cdot\cdot\cdot (p^m - p^{m-1})$, and is simply isomorphic with the direct product of a cyclic group of order $p-1$ and a group formed by extending an abelian group of order p^m, type $(1, 1, \cdots, 1)$ by all those operators from its own group of isomorphisms which leave invariant the operators of one its own group of isomorphisms which leave invariant the operators of one cyclic group of order p, and then multiplying this extended group by an operator of order p^{n-1} which is commutative with each operator of the extended group and which has one of the invariant operators of order p for its p^{n-2}th power.*

This shows that *for a given odd prime p and a fixed value of $m > 0$, the group of isomorphisms of each abelian group of the system $(n = (2), 3, 4, \cdots,)$ contains the group of isomorphisms of the preceding as an invariant subgroup of index p,* since they differ only in the order of the operator by which the extended group is multiplied.

Again, since multiplying the extended group by the designated operator of order p^{n-1} is equivalent to taking an abelian group of order p^{m+n-2}, type $(n-1, 1, \cdots, 1)$ and extending it by all those operators from its own group of isomorphisms that leave invariant the operators of exactly one of its subgroup of order p^{n-1}, the preceding theorem can be stated as follows, and from it can be seen that *the group of isomorphisms of each abelian group of the system under study is an extension of the one of the system just before it and of index p in it.*

Theorem 3′. *The I of an abelian group G of order p^{m+n-1}, type $(n, 1, 1, \cdots, 1)$, p an odd prime and $n > 1$, is of order $(p-1)p^{m+n-2}(p^m - p)$*

* Cf. Burnside, *loc. cit.*, § 48.

$(p^m - p^2) \cdots (p^m - p^{m-1})$ and is simply isomorphic with the direct product of a cyclic group of order $p-1$ and the group formed by extending an abelian group of order p^{m+n-2}, type $(n-1, 1, \cdots, 1)$ by all those operators from its own group of isomorphisms that leave invariant the operators of exactly one of its cyclic subgroups of order p^{n-1}.

If p is the even prime and $n > 2$, F is not cyclic but is an abelian group of order 2^{n-1}, type $(n-2, 1)$,* and the 2^{n-3}th power of the operators of order 2^{n-2} generates the cross-cut of F and E, a group of order two. Accordingly, the counterparts of the preceding two theorems are:

Theorem 4. *The I of an abelian group G of order 2^{m+n-1}, type $(n, 1, 1, \cdots, 1)$, $n > 2$, is of order $2^{m+n-2}(2^m - 2)(2^m - 2^2) \cdots (2^m - 2^{m-1})$ and is simply isomorphich with the direct product of a group of order 2 and a group formed by extending an abelian group of order 2^m, type $(1, 1, \cdots, 1)$ by all those operators from its own group of isomorphisms which leave invariant one operator of order two, and then multiplying this extended group by an operator of order 2^{n-2} which is commutative with each operator of the extended group and which has the invariant operator of order two for its 2^{n-3}th power.*

This and the following equivalent statement of the proposition show the inclusive relation between the groups of isomorphisms of two consecutive groups of the system:

Theorem 4. *The I of an abelian group of order 2^{m+n-1}, type $(n, 1, 1, \cdots, 1)$, $n > 2$, is of order $2^{m+n-2}(2^m - 2)(2^m - 2^2) \cdots (2^m - 2^{m-1})$ and is simply isomorphic with the direct product of a group of order 2 and the group formed by extending an abelian group of order 2^{m+n-3}, type $(n-2, 1, \cdots, 1)$ by all those operators from its own group of isomorphisms that leave invariant individually the operators of exactly one of its cyclic subgroups of order 2^{n-2}.*

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* Burnside, *loc. cit.*, § 169.

On the Satellite Line of the Cubic.

By R. M. Winger.

The Rational Cubic.

1. *Introduction.* While Salmon * discusses the satellite line of the cubic at some length he fails to give its equation. The equation was supplied by Cayley † who calculated it laboriously by a direct method. Several covariant expressions for this remarkable line were given by Walker.‡ And another has been furnished by Morley.§ In the present paper the explicit equation of the satellite is exhibited both for the rational and the general cubic, in canonical form, and some associated loci are considered. Several interesting chain theorems are obtained and a generalization is made for the plane curve of order n.

We consider first the rational cubic R_4^3, taking as its equations in points and lines respectively

$$x_0 = 3t^2, \quad x_1 = 3t, \quad x_2 = t^3 + 1, \tag{1}$$

or

$$x_0^3 + x_1^3 - 3x_0x_1x_2 = 0,$$

$$u_0 = 1 - 2t^3, \quad u_1 = t^4 - 2t, \quad u_2 = 3t^2, \tag{2}$$

or

$$u_2^4 - 6u_0u_1u_2^2 - 3u_0^2u_1^2 + 4u_0^3u_2 + 4u_1^3u_2 = 0,$$

when the triangle of reference is the invariant triangle of the G_6 which leaves the curve unaltered.

An important curve for the satellite theory is the conic N, || the locus of lines joining pairs of contacts of tangents from points of the R_4^3. The equation of N for the cubic (1) is

$$u_2^2 - 9u_0u_1 = 0. \tag{3}$$

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* *Higher Plane Curves*, third edition, Art. 149 ff.

† "A Memoir on Curves of the Third Order," *Phil. Trans.* (1857), p. 439, or *Collected Papers*, Vol. II, p. 405.

‡ *Phil. Trans.* (1888) A, p. 170 and *Proc. Lond. Math. Soc.*, Vol. XXI (1890), p. 247.

§ University lectures for the session 1910-1911.

|| Morley, American Journal of Mathematics, Vol. XI (1889), p. 316; Winger, "Involutions on the Rational Cubic," *Bulletin, Amer. Math. Soc.*, October, 1918, p. 28. This conic touches the nodal tangents where they meet the line of flexes and has contacts with the curve at the sextactic points.

While every line u' has a unique satellite u, a line u is in general satellite to four lines u'. Continuing the figure we shall call the line u' the *primary* of u. Combining these statements we have the theorem: *The 6 contacts of tangents from 3 collinear points (on u) of a rational cubic lie by threes on the 4 sides u' of a quadrilateral whose diagonal 3-line is circumscribed to N.*

When u' is tangent to the curve, the satellite is likewise tangent, at the point where u' cuts again. If however u is tangent two of the primary lines coincide. Thus the tangent at a point is satellite to the two tangents from the point and the line, counted twice, joining contacts of those tangents.

2. The line $u' \equiv (ux) \equiv u_0x_0 + u_1x_1 + u_2x_2 = 0$

cuts the curve in three points with parameters

$$u_2t^3 + 3u_0t^2 + 3u_1t + u_2 = 0. \tag{4}$$

If s_i refer to the symmetric functions of these t's and σ_i to the parameters $(-1/t^2)$ of the three tangential points we have

$$s_1 = -3u_0/u_2, \quad s_2 = 3u_1/u_2, \quad s_3 = -1, \tag{5}$$
$$\sigma = -\Sigma\, 1/t_i^2 = -(2s_1 + s_2^2), \quad \sigma_2 = s_1^2 - 2s_2, \quad \sigma_3 = -1.$$

Whence the equation giving the tangential points is

$$t^3 + (2s_1 + s_2^2)t^2 + (s_1^2 - 2s_2)t + 1 = 0 \tag{6}$$

and the equation of the line u on which they lie is

$$\Sigma_2\colon\ (3u_1^2 - 2u_0u_2)x_0 + (3u_0^2 - 2u_1u_2)x_1 + u_2^2x_2 = 0. \tag{7}$$

For given u' (7) is the equation of the satellite line. Equally for given x and variable u it is the equation of a conic, the locus of lines whose satellites pass through x. When considered as a conic we shall denote (7) by Σ_2.

If x is on the curve the conic degenerates to the two contacts of tangents from x. Hence the discriminant of the conic is to within a factor the point equation of the R_4^3.

3. If x is not on the cubic 4 tangents u_i can be drawn to the curve. From each point of contact t_i can be drawn 2 tangents u'_{i1} and u'_{i2}, with contacts t'_{i1} and t'_{i2}. These tangents u' and the line u'_{in} joining their contacts are all primaries of u_i. *Hence the four pairs of tangents u'_{i1}, u'_{i2} which can be drawn to the R_4^3 from the contacts t_i of tangents from a point x, together with the four lines u'_{in} joining contacts t'_{i1}, t'_{i2} of the pairs touch a conic Σ_2. The four pairs of tangents are the common lines of conic and R_4^3 and the four lines u'_{in} are the common lines of Σ_2 and conic N.*

Continuing, from each of the 8 points t' can be drawn a pair of tangents u'' with contacts t'' and these contacts determine a line u''_n. *These* 24 *lines all touch a curve of class four, viz. the locus of lines whose satellites envelop the conic* Σ_2. *The* 16 *lines* u'' *are the common lines of the quartic and the* R_4^3 *while the* 8 *lines* u''_n *are the common lines of the quartic and the conic* N.

Starting with a curve Σ_1 of class one, namely the point x, we have thus a chain of theorems on class curves associated with the rational cubic for the process can be continued indefinitely. This chain is however a special case of another which can be obtained by direct generalization of the theorem implied by equation (7). Since the equation of the satellite line is of the second degree in the coefficients of the primary, we may say: *If the satellite line* u *run around a curve* Σ_δ *of class* δ, *the primary* u' *envelops a curve* $\Sigma_{2\delta}$ *of class* 2δ. *The equation of* $\Sigma_{2\delta}$ *can be found at once by replacing* u_0, u_1, u_2 *in the homogeneous line equation of* Σ_δ *by*

$$(3u_1^2 - 2u_0u_2) : (3u_0^2 - 2u_1u_2) : u_2^2$$

respectively.

Now the R_4^3 and Σ_δ have 4δ common lines u. If the contacts be designated by t, then from each t can be drawn two tangents u' to the cubic. The contacts t' of these tangents determine a line u'_n. These lines u' are the three primaries of u and hence are lines of $\Sigma_{2\delta}$. *We have thus* 12δ lines of $\Sigma_{2\delta}$,—*the* 8δ *lines* u' *which are the common lines of* $\Sigma_{2\delta}$ *and* R_4^3 *and the* 4δ *lines* u'_n *which the common lines of* $\Sigma_{2\delta}$ *and the conic* N. From the 8δ points t' can be drawn in turn 8δ pairs of tangents u'' which determine 8δ lines u''_n joining contacts t'' of the pairs. *These lines all touch a curve* $\Sigma_{4\delta}$ *of class* 4δ, *the locus of lines whose satellites envelop* $\Sigma_{2\delta}$. *The* 16δ *lines* u'' *are the common lines of* $\Sigma_{4\delta}$ *and the cubic and the* 8δ *lines* u''_n *are the common lines of* $\Sigma_{4\delta}$ *and the conic* N. *And so on ad infinitum.* Hence any curve Σ_δ establishes a chain which reduces to the old when $\delta = 1$.

4. We come now to the converse problem: given a class curve to find the satellite locus. Theorem. *If a line touch a curve* S_δ *of class* δ, *the satellite envelops a curve* $S_{2\delta}$ *of class* 2δ.* *In particular if* S_δ *is rational so also is* $S_{2\delta}$. If we denote the satellite locus by S_μ, then we want the number of lines of S_μ which pass through an arbitrary point. Now those lines u' of S_δ whose satellites pass through a given x touch a conic Σ_2. But S_δ and the conic have just 2δ lines in common i. e. there are 2δ lines of S_μ on x and $\mu = 2\delta$.

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* This involves no contradiction with the theorem of the previous section; for each line u has four primaries u', hence as u' runs around $\Sigma_{2\delta}$, u will generate an $S_{4\delta}$, which is Σ_δ repeated four times.

To prove the second part of the theorem, let

$$u_0(t)x_0 + u_1(t)x_1 + u_2(t)x_2 = 0, \tag{8}$$

where the u's are binary forms of order δ be the map equation in lines of a rational curve R_δ of class δ. The map equation of the satellite locus is found by replacing the coefficients of the x's by

$$(3u_1^2 - 2u_0u_2) : (3u_0^2 - 2u_1u_2) : u_2^2$$

respectively. The result is obviously an equation in which the coefficients of the x's are rational and of degree 2δ in t.

To find the common lines of $S_{2\delta}$ and R_4^3 it is only necessary to recall that the satellite is tangent to the cubic only if the primary is a tangent of the cubic or a line of conic N. S_δ and R_4^3 have 4δ common lines whose satellites, viz. the tangents at the points in which the lines cut R_4^3 again, account for 4δ common lines of R_4^3 and $S_{2\delta}$. The 2δ primaries which touch the conic N therefore must take up the remaining 4δ common lines, i. e. each of these satellites will count for two.

As an application of these theorems we observe

(*a*) *The satellite locus of the conic N is the cubic itself.*

(*b*) *The cubic taken twice is its own satellite locus.*

(*c*) *The locus of lines whose satellites touch the cubic is a composite curve of class eight consisting of the cubic and the conic N repeated.*

THE NON-SINGULAR CUBIC.

5. Most of the theorems stated for the rational cubic can be extended at once to the case of the general cubic C_6^3. The tangent at a point P is now satellite to 10 lines, viz. the 4 tangents from the point and the 6 lines (repeated) joining in pairs the contacts of these tangents. Salmon * has remarked that the envelope of the 6 lines for variable P is a certain composite curve M_9 consisting of three class cubics.

Morley † has shown that the satellite of the line (ux) is

$$\{\Omega^2 - 1/6\,(ux)\Omega^3\}\, C^3\Gamma_6 \tag{9}$$

where C^3 and Γ_6 are respectively the point and line equations of the curve and Ω is the ternary differential operator

$$\left(\frac{\partial^2}{\partial x_0 \partial u_0} + \frac{\partial^2}{\partial x_1 \partial u_1} + \frac{\partial^2}{\partial x_2 \partial u_2}\right)$$

* *Higher Plane Curves*, Art. 151.

† Loc. cit.

Calculated for the canonical form

$$x_0^3 + x_1^3 + x_2^3 + 6ax_0x_1x_2 = 0, \tag{10}$$

the equation of the satellite line is

$$\Gamma_4 : \sum_0^2 \{u_i^4 - 2(u_j^3 + u_k^3)u_i - 6a\, u_j^2u_k^2\}x_i = 0, \quad i \neq j \neq k \tag{11}$$

Again for given x equation (11) represents a class quartic Γ_4, the locus of lines whose satellites are on x. This curve is of the Humbert or *desmic* type.* A line quartic is desmic if it belongs to a pencil of quartics which contains three degenerate curves each composed of four points, the three degenerate curves then being desmic quadrangles. That Γ_4 is desmic can be shown as follows. Consider a line u cutting the cubic in points a, b and c. Denote the contacts of tangents from these points by α_i, β_i, γ_i, $(i = 1, 2, 3, 4)$. Any point x on u will determine a curve Γ_4. Two points x and x' thus determine a pencil

$$\Gamma_4 + \lambda\Gamma'_4 \tag{12}$$

of curves associated with the points $x + \lambda x'$ of u. Now the primary lines of the (line) pencil on a are obviously the four (line) pencils on points α_i. In other words points α constitute a degenerate member of the pencil (12). This pencil of quartics thus contains the three desmic quadrangles α, β, γ which proves the theorem.

We have the following theorems for the general cubic, omitting the proofs which follow closely those given at length for the earlier case.

If the satellite line run around a curve Γ_δ of class δ, the primary envelops a curve $\Gamma_{4\delta}$ of class 4δ whose equation is obtained by replacing u_0, u_1, u_2 in the homogenous equation of Γ_δ by the coefficients of the x's in equation (11). Γ_δ and C_6^3 have 6δ common lines u with contacts s. From each point s can be drawn to the cubic 4 tangents u' with contacts s'. Joining the points s' in pairs are 6 lines u'_m. There are thus 60δ tangents of $\Gamma_{4\delta}$ which comprise the primaries of lines u,—the 24δ tangents u' which are the common lines of $\Gamma_{4\delta}$ and the cubic, and the 36δ lines u'_m which are the common lines of $\Gamma_{4\delta}$ and the curve M_9.

This is the first link in a chain of theorems associated with every curve Γ_δ. A chain of especial interest is that originating with a point Γ_1:

* After Humbert who discusses such curves in two papers, *Journal de Mathématiques*, 4e Série, Vol. VI (1890), p. 423 and VII (1891), p. 353. Professor Morley called my attention to the fact that Γ_4 is desmic. Since it depends on eleven constants it would appear to be the general desmic quartic.

From a point x can be drawn in general 6 tangents u with contacts s. From each point s can be drawn 4 tangents u' with contacts s' and the points s' can be joined in pairs by 6 lines u'_m. The 24 tangents u' and 36 lines u'_m touch a curve Γ_4 of class four, namely the locus of lines whose satellites pass through x. The tangents u' are the common lines of Γ_4 and C_6^3 while the lines u'_m are the common lines of Γ_4 and M_9. Again the 24 points s' determine 96 tangents u'' with contacts s'' and 144 lines u''_m joining these contacts in pairs. These 240 lines u'' touch a curve Γ_{16}, of class 16, the primary locus of Γ_4, and comprise the lines which Γ_{16} has in common with C_6^3 and M_9. And so on forever.

Most of the peculiarities of Humbert's curve as summarized (for the dual) in section 12 of his first paper can be recovered readily with our present apparatus. Indeed it seems preferable to reverse his procedure and derive the properties of the quartic from the cubic and point which define it.

The six characteristic 4-points whose diagonal points are nodes and whose connecting lines are the nodal tangents are the six sets of points s' in the theorem just stated. In other words, *the 36 intersections of Γ_4 and the cubic are at these 18 nodes and the common lines of Γ_4 and M_9 are the 36 nodal tangents,—each component of M_9 touching the tangent pair of one node in each set.* Or again *from the tangentials of the points s can be drawn 18 tangents (in addition to the lines u), the contacts of which are at the 18 nodes.*

Likewise the satellite locus of a curve C_δ of class δ is a curve $C_{4\delta}$ of class 4δ rational if C_δ is rational. The common lines of $C_{4\delta}$ and C_6^3 are (a) the satellites of the 6δ common lines of C_δ and the cubic and (b) the satellites of the 9δ common lines of C_δ and M_9, the latter lines each counting for two.

The satellite locus of M_9 is the cubic repeated six times.

The satellite locus of the cubic is the curve itself four times repeated; while the primary locus consists of the cubic and M_9 repeated.

Since M_9 is the double curve in the primary locus of the cubic it must be the Jacobian of the coefficients in the satellite line.* Calculated thus and verified for special lines the equation of M_9 is found to be

$$U^3 - 36a^2U^2 - 54aU - 54(1 + 4a^3) = 0 \tag{13}$$

where
$$U = \frac{u_0^3 + u_1^3 + u_2^3}{u_0u_1u_2}.$$

Hence the factors of M_9 are

$$u_0^3 + u_1^3 + u_2^3 - k_iu_0u_1u_2 = 0, \tag{14}$$

* Likewise the Jacobian of the coefficients in the satellite (7) of the rational cubic is Nu_2 which incidentally proves the defining property of N.

where k_i are the roots of (13) considered as a cubic in U. It follows that the three components belong to the syzygetic pencil of class cubics determined by the nine harmonic polars.

M_9 also touches the cubic at the 27 sextactic points. Hence the tangents to the cubic from the flexes account for all the common lines of the two curves.

THE GENERAL PLANE CURVE.

6. Consider now the plane curve C_δ^n of order n and class δ. The tangents at n points of a line (ux) constitute a curve T^n which meets C^n in n^2 points. Since $2n$ of these points are the complete intersections of C^n and a curve of order 2,—the line repeated,—it follows from the theory of residuation that the remaining $n(n-2)$ lie on a C^{n-2}, the satellite $(n-2)$-ic of the line u.

The equation of the satellite curve, $f(u^k x^{n-2}) = 0$, will contain u as well as x. For a given x therefore, $f = 0$ represents a curve of class k, the locus of lines whose satellites pass through x. To ascertain the value of k, it will be sufficient to enumerate the common lines of C_μ^n and f considered as a class curve. Let u' be such a line. Then since u' is a tangent to C^n its satellite curve degenerates into the $n-2$ tangents at the remaining intersections of u' and C^n. One of these $n-2$ lines must pass through x in virtue of the defining property of f, i. e. u' is a tangent to C^n from a contact of one of the tangents from x. Since C_δ^n is of class δ, there are precisely $\delta(\delta-2)$ lines u'. Hence $k = \delta - 2$.*

We have at once the following generalization of the first link of the special chain theorem for the cubic. *Denoting by $t_1, t_2, \cdots\cdots t_\delta$ the contacts of the δ tangents from an arbitrary point x to a curve C_δ^n of order n and class δ, the δ sets of $(\delta-2)$ tangents from points t_i comprise the common lines of C_δ^n and a curve f of class $(\delta-2)$, namely the locus of lines whose satellite $(n-2)$-ics pass through x.* If C_δ^n is non-singular f is of class $(n+1)(n-2)$ and there are $(n+1)(n)(n-1)(n-2)$ common lines.

UNIVERSITY OF OREGON,
January, 1918.

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* It can be shown by the analytic method employed by Morley, l. c. that f involves the coefficients of C_δ^n to the degree $2n-1$. For an application of that method when $n=4$, see a paper on the satellite conic of the quartic by T. Cohen, AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXVIII (1916), p. 325.